

Generic family displaying robustly a fast growth of the number of periodic points

Pierre Berger*

January 11, 2017

Abstract

For any $2 \leq r \leq \infty$, $n \geq 2$, we prove the existence of an open set U of C^r -self-mappings of any n -manifold so that a generic map f in U displays a fast growth of the number of periodic points: the number of its n -periodic points grows as fast as asked. This complements the works of Martens-de Melo-van Strien, Gochenko-Shil'nikov-Turaev, Kaloshin, Bonatti-Díaz-Fisher and Turaev, to give a full answer to questions asked by Smale in 1967, Bowen in 1978 and Arnold in 1989, for any manifold of any dimension and for any smoothness.

Furthermore for any $2 \leq r < \infty$ and any $k \geq 0$, we prove the existence of an open set \hat{U} of k -parameter families in U so that for a generic $(f_a)_a \in \hat{U}$, for every $\|a\| \leq 1$, the map f_a displays a fast growth of periodic points. This gives a negative answer to a problem asked by Arnold in 1992 in the finitely smooth case.

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*CNRS-LAGA, Université Paris 13, USPC

Given a diffeomorphism (or a local diffeomorphism) f of a compact manifold M , we denote by $Per_n f := \{x \in M : f^n(x) = x\}$ the set of its n -periodic points. To study its cardinality, we consider also the subset $Per_n^0 f \subset Per_n f$ of isolated n -periodic points. We notice that the cardinality of $Per_n^0 f$ is an invariant by conjugacy. Hence it is natural to study the growth of this cardinality with n .

Clearly, if f is a polynomial map, the cardinality $Per_n^0 f$ is bounded by the degree of f^n , which grows at most exponentially [DNT16].

The first study in the C^∞ -case goes back to Artin and Mazur [AM65] who proved that there exists a dense set \mathcal{D} in $Diff^r(M)$, $r \leq \infty$, so that for every $f \in \mathcal{D}$, the number $Card Per_n^0 f$ grows at most exponentially, i.e. , there exists $K(f) > 0$ so that:

$$(A-M) \quad \frac{1}{n} \log Card Per_n^0 f \leq K(f) .$$

This leads Smale [Sma67] and Bowen [Bow78] to wonder about the relationship between rate of growth of the number of periodic points on the one hand and dynamical ζ -function or topological entropy on the other hand for (topologically) generic diffeomorphisms. In particular, these questions asked whether (A-M) diffeomorphisms are generic. Finally Arnold asked the following problems:

Problem 0.1 (Smale 1967, Bowen 1978, Arnold Pb. 1989-2 [Arn04]). *Can the number of fixed points of the n^{th} iteration of a topologically generic infinitely smooth self-mapping of a compact manifold grows, as n increases, faster than any prescribed sequence $(a_n)_n$ (for some subsequences of time values n)?*

We recall that a property is *topologically generic* if it holds for a countable intersection of open and dense sets. The topology on the space of C^∞ -maps is the union of the ones induced by the C^r -topologies $C^r(M, M)$ among $r \geq 0$ finite.

Another notion to quantify the abundance of a phenomena was sketched by Kolmogorov during his plenary talk at the ICM in 1954. Then his student Arnold formalized it as follows [IL99, KH07]:

Definition 0.2 (Arnold's Typicality). *A property (\mathcal{P}) on the set of C^r -mappings $C^r(M, M)$ of M is typical if for every $k \geq 1$, for a topologically generic C^r -family of $(f_a)_{a \in \mathbb{R}^k}$ of C^r -maps f_a , for Lebesgue almost every $a \in \mathbb{R}^k$, the map f_a satisfies the property (\mathcal{P}) .*

We recall that $(f_a)_{a \in \mathbb{R}^k}$ is of class C^r if the map $(a, z) \mapsto f_a(z)$ is of class C^r . When $r < \infty$, the topology on this space is equal to the compact-open C^r -topology of $C^r(\mathbb{R}^k \times M, M)$. When $r = \infty$, the topology on the space of C^∞ -families is the one given by the union of those induced by the $C^{r'}$ -topologies among $r' \geq 0$ finite.

Problem 0.3 (Arnold 1992-13 [Arn04]). *Prove that a typical, smooth, self-map f of a compact manifold satisfies that $(Card Per_n f)_n$ grows at most exponentially fast.*

Remark 0.4. Many other Arnold's problems are related to this question [Arn04, 1994-47, 1994-48, 1992-14].

These problems enjoy a long tradition.

In dimension 1, Martens-de Melo-van Strien [MdMvS92] showed that for every $\infty \geq r \geq 2$, for an open and dense set¹ of C^r -maps, the number of periodic points grows at most exponentially fast.

¹whose complement is the infinite codimensional manifold formed by maps with at least one flat critical point.

Kaloshin [Kal99] answered to a question of Artin and Mazur (in the finitely smooth case) by proving that for a dense set \mathcal{D} in $Diff^r(M)$, $r < \infty$, the set $Per_n f$ is finite for every n (and so equal to $Per_n f$) and its cardinality grows at most exponentially fast.

However, Kaloshin [Kal00] proved that whenever $2 \leq r < \infty$ and $\dim M \geq 2$, a *locally topologically generic diffeomorphism displays a fast growth of the number of periodic points*: there exists an open set $U \subset Diff^r(M)$, so that for any sequence of integers $(a_n)_n$, a topologically generic $f \in U$ satisfies:

$$(\star) \quad \limsup_{n \rightarrow \infty} \frac{Card P_n^0(f)}{a_n} = \infty.$$

Furthermore, Bonatti-Diaz-Ficher [BDF08] extended this result to the C^1 -case in dimension ≥ 3 . The conservative counterpart of this result in the conservative case has been proved by Kaloshin-Saprykina in [KS06]. In Gochenko-Shilnikov-Turaev [GST93b], Kaloshin theorem is based on a result by Gonchenko-Shilnikov-Turaev [GST93b, GST99] that surface diffeomorphisms with degenerate parabolic points form a locally dense subset of $Diff^r(M^2)$; since their proof in [GST99, GST07] is valid in the C^∞ case, Kaloshin theorem immediately extends to the C^∞ -case as well. Recent seminal work of Turaev [Tur15] also implies that among C^∞ -surface diffeomorphisms the fast growth of the number of periodic points is locally a topologically generic property.

However the C^∞ -case in dimension ≥ 3 remained open². In this term, our first result accomplishes the study of problem 0.1, in any smoothness ≥ 2 and any dimension:

Theorem A. *Let $\infty \geq r \geq 2$ and let M be a compact manifold of dimension n .*

If $n = 1$, Property (AM) is satisfied by an open and dense set of C^r -self-mapping.

If $n \geq 2$, there exists a (non-empty) open set $U \subset Diff^r(M)$ so that given any sequence $(a_n)_n$ of integers, a topologically generic f in U satisfies (\star) .

Actually, the proof of this theorem will be done in dimension ≥ 3 for the diffeomorphism case, and in dimension 2 for self-mappings. For the one dimensional case has been proved in [MdMvS92] whereas the surface diffeomorphism case is proved as aforementioned.

This result will be proved following a method which contains one aspect related to the work Asaoka-Shinohara-Turaev [AS^Tar] on the fast growth of the number of periodic points for a locally generic free group action of the interval. In our proof, basically a free group of diffeomorphisms of the circle is embedded into the manifold as a normally hyperbolic fibration by circles. As in Asaoka-Shinohara-Turaev's approach, we consider a robust hetero-dimensional cycle given by a Bonatti-Diaz Blender [BD96]. Thanks to a new renormalization trick, we exhibit a dense set of perturbations which display a parabolic dynamics on an invariant, finite union of circles. We perturb it to a rotation thanks to Herman's development of KAM-theorem. Then it is easy to construct a topologically generic perturbation which exhibits a fast growth of the number of periodic points.

As there are topologically generic sets of the real line whose Lebesgue measure is null, a negative answer to Problem 0.1 does not need to suggest a negative answer to Problem 0.3.

To provide a positive answer to Arnold's Problem 0.3, Hunt and Kaloshin [KH07, KH] used a method described in [GHK06] to show that for $\infty \geq r > 1$, a *prevalent* C^r -diffeomorphisms

²In [GST93a], Theorem 7, Gonchenko-Shilnikov-Turaev theorem was claimed to be true for any dimension but no proof has ever been published.

satisfies:

$$(\diamond) \quad \limsup_{\infty} \frac{\log P_n(f)}{n^{1+\delta}} = 0, \quad \forall \delta > 0.$$

The notion of prevalence was introduced by Hunt, Sauer and York [HSY92]. A property is *prevalent* if *roughly speaking* almost all perturbations in the embedding of a Hilbert cube at every point of a Banach space (like $C^r(M, M)$), the property holds true. We notice that (\diamond) is satisfied for a prevalent diffeomorphism but not for a topologically generic diffeomorphism (see other examples of mixed outcome in [HK10]).

However the latter did not completely solve Arnold's problem 0.3 in particular because the notion of prevalence is *a priori* independent to the notion of typicality initially meant by Arnold. Indeed his problem was formulated for typicality in the sense of definition 0.2 (see explanation below problem 1.1.5 in [KH07]).

In this term the second and main result of this work is surprising since it provides a negative answer to Arnold's problem 0.3 in the finitely smooth case:

Theorem B. *Let $\infty > r \geq 1$ and $0 \leq k < \infty$, let M be a manifold of dimension ≥ 2 .*

Then there exists a (non-empty) open set \hat{U} of C^r -families $(f_a)_a$ of C^r -self-mappings f_a of M so that, for any sequence of integers $(a_n)_n$, a topologically generic $(f_a)_a \in \hat{U}$ consists of map f_a satisfying (\star) , for every $\|a\| \leq 1$. Moreover if $\dim M \geq 3$, we can chose \hat{U} to be formed by families of diffeomorphisms.

Remark 0.5. Actually, the same proof shows that the statement of Theorem B holds true in the category of C^r -family of C^∞ -self-mappings.

The proof of this theorem follows the same scheme as for Theorem A, beside the fact that the blender is replaced by a new object: the λ -parablender (which generalizes both the blender and the parablender as introduced in [Ber16b]). A generalization to the parameter case of the renormalization trick enables us to display a dense set of families of self-mappings which leave invariant a finite union of normally hyperbolic circles on which the restrictions are constantly parabolic. Then a careful study of the parabolic bifurcation and renormalization together with KAM-Herman-Yoccoz' Theorem enable us to perturb these families by one which exhibits a constant family of rotations. Finally it is easy to perturb the family so that it displays a fast growth of the number of periodic points at every parameter $\|a\| \leq 1$.

This final step (based on an application of the KAM Theorem) has been implemented recently by Asaoka [Asa16] who showed that for every $r \in \{\infty, w\}$, there exists a C^r -open set of *conservative* surface diffeomorphisms in which typically in the sense of Arnold, a map displays a fast growth of the number of periodic points.

To conclude this introduction, let me recall that Arnold's philosophy was not to propose *problems of binary type admitting a "yes-no" answer*, but rather to propose *wide-scope programs of explorations of new mathematical (and not only mathematical) continents, where reaching new peaks reveals new perspectives, and where a preconceived formulation of problems would substantially restrict the field of investigations that have been caused by these perspectives. [...]* *Evolution is more important than achieving records*, as he explained in his preface [Arn04].

Let us remark that in this sense the contrast between the result of Kaloshin-Hunt and the main result of this manuscript is interesting since they shed light on how an answer to a question might depend on the definition of typicality.

Furthermore, the proofs of this work do not only answer questions, they also develop new tools which will certainly be useful for our program on emergence [Ber16a]. Let us notice that the C^∞ -case of Problem 0.3 (or conjecture 1994-47 [Arn04]) remains open, although in view of Remark 0.5, I would bet for a negative answer; I would even dare to propose:

Conjecture 0.6. *For every $r \in \{1, \dots, \infty, \omega\}$, there exists an open set of diffeomorphisms $U \in \text{Diff}^r(M)$, so that given any $k \geq 0$, for any C^r -generic family $(f_a)_{a \in \mathbb{R}^k}$ with $f_a \in U$, for every a small, the growth of the number of periodic points of f_a is fast.*

I am thankful to Abed Bounemoura, Hakan Eliason, Bassam Fayad, Vadim Kaloshin and Rafael Krikorian for our interesting conversations. I am grateful to Sylvain Crovisier and Enrique Pujals for our inspiring discussions and Dmitry Turaev for his important comments on the parabolic renormalization for circle diffeomorphisms.

1 Concepts involved in the proofs

We shall first recall some elements of one dimensional dynamics, and more specifically the concept of parabolic map and the KAM-Herman theorems.

Then we will recall some elements of hyperbolic theory, including the concepts of blender and a new feature, the λ -blender which will be useful to construct a dense set of parabolic maps on normally hyperbolic embedded circles.

Afterward we will generalize these concepts to parameter families. The concepts of blender and λ -blender will be generalized to the C^r -parablender and a new object, the C^d - λ -parablender.

Finally we will recall the Hirsch-Pugh-Shub Theory [HPS77], and its extension [Ber10] for the endomorphisms case, for they will be useful for the proof of the theorems.

1.1 Dynamics of the circle

Given a homeomorphism $g \in \text{Diff}^0(\mathbb{R}/\mathbb{Z})$ of the circle \mathbb{R}/\mathbb{Z} , one defines its rotation number ρ_g as follows. We fix $G \in \text{Diff}^0(\mathbb{R})$ a lifting of g for the canonical projection $\pi: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$:

$$\pi \circ G = g \circ \pi .$$

Then Poincaré proved that $\rho_G = \lim_{n \rightarrow \infty} G^n(0)/n$ is uniquely defined. Furthermore, $\rho_g = \pi(\rho_G)$ does not depend on the lifting G of g . The *rotation number* of g is ρ_g .

It is easy to show that the rotation number depends continuously on g .

1.1.1 Maps with Diophantine rotation number

A number $\rho \in \mathbb{R}$ is *Diophantine*, if there exist $\tau > 0$ and $C > 0$ so that for every $p, q \in \mathbb{N} \setminus \{0\}$ it holds:

$$|q\rho - p| \geq Cq^{-\tau} .$$

Let us recall that the set of Diophantine numbers is of full Lebesgue measure.

Here is an improvement of Yoccoz of Herman's theorem of Arnold Conjecture:

Theorem 1.1 (Arnold-Herman-Yoccoz [Her79, Yoc84]). *If the rotation number of $g \in \text{Diff}^\infty(\mathbb{R}/\mathbb{Z})$ is a Diophantine number ρ , then there exists $h \in \text{Diff}^\infty(\mathbb{R}/\mathbb{Z})$ which conjugates g with the rotation R_ρ of angle ρ :*

$$h \circ g \circ h^{-1} = R_\rho .$$

Moreover, if $(g_a)_a$ is a C^∞ -family of diffeomorphisms with constant rotation number ρ which is Diophantine, then there exists a C^∞ -family $(h_a)_a$ of diffeomorphisms h_a which conjugates $(g_a)_a$ with R_ρ :

$$h_a \circ g_a \circ h_a^{-1} = R_\rho .$$

Proof. The first part of this theorem is the main theorem of [Yoc84]. Since the conjugacy is uniquely defined up to a composition with a rotation, the second part of this Theorem is a local problem. Hence by the first part, it suffices to show that if $(g_a)_a$ satisfies moreover $g_0 = R_\rho$, then the conjugacy $(h_a)_a$ can be chosen smooth on a neighborhood of $a = 0$. This is a direct consequence of Theorem 3.1.1 of [Bos85]. \square

Hence once we exhibit an invariant C^∞ -circle on which the dynamics displays a diophantine rotation number ρ , by the above theorem, we exhibit an invariant circle on which f acts as a rotation of angle ρ . Then it is easy to perturb this dynamics of the circle to a root of the identity (by moving ρ to a rational number). Finally we will show that it is easy to perturb a root of the identity to a diffeomorphism with a large number of hyperbolic periodic points.

1.1.2 Parabolic maps of the circle

A key new idea in this work is to exhibit circle diffeomorphisms with Diophantine rotation number by creating first *parabolic diffeomorphisms of the circle* (they are indeed easier to exhibit densely thanks to geometrical arguments).

Definition 1.2. A C^2 -diffeomorphism g of a circle \mathbb{T} is parabolic if there exists $p \in \mathbb{T}$ so that

- The point p is the unique fixed point of g ,
- The point p is a non-degenerated parabolic fixed point of g :

$$g(p) = p, \quad D_p g = 1 \quad , \quad D_p^2 g \neq 0 \quad .$$

This idea might sound anti-intuitive since the rotation number of a parabolic map is zero.

The interest of parabolic maps of the circle is that they have a geometric definition and produce irrational rotations after perturbations. Indeed if g is a C^r -parabolic circle map, for $r \geq 2$, then its rotation number is 0. Also one sees immediately that the composition $R_\epsilon \circ g$, with R_ϵ a rotation of angle $\epsilon > 0$ small, has non-zero rotation number. Hence, by continuity of the rotation number and density of Diophantine number in \mathbb{R} , we can choose $\epsilon > 0$ arbitrarily small so that the rotation number $\rho(\epsilon)$ of $R_\epsilon \circ g$ is Diophantine. This proves:

Proposition 1.3. For every $r \geq 2$, the set D^r of C^r -circle maps with Diophantine rotation number accumulates on the set P^r of C^r -parabolic maps:

$$cl(D^r) \supset P^r .$$

The above argument is topological. Hence the following is a non trivial extension of the latter proposition for parameter family, proved in section 5:

Theorem 1.4. Let $k \in \mathbb{N}$ and let $V' \Subset V \subset \mathbb{R}^k$ be open subsets. Given any C^∞ -family $(g_a)_{a \in V}$ of circle maps so that for every $a \in V$ the map g_a is parabolic, there exists an arbitrarily small Diophantine number $\alpha > 0$, there exists a small C^∞ -perturbation $(g'_a)_a$ of $(g_a)_a$ so that:

- the rotation number of g'_a is α for every $a \in V'$.

- the family $(g'_a)_{a \in V}$ is of class C^∞ .

The proof will involve the parabolic renormalization for an unfolding of $(g_a)_a$, and the Arnold-Herman Theorem.

1.2 Hyperbolic theory

Most of the arguments will be done first for surface local diffeomorphisms. We recall that a map $f \in C^r(M, M)$ is a local diffeomorphism, if its derivative is everywhere bijective.

Let us recall some elements of the hyperbolic theory for local diffeomorphisms, proofs are given in [BR13, Ber10].

An invariant compact set K for f is *hyperbolic* if there is a vector bundle $E^s \subset TM|_K$ which is invariant by $Df|_K$, contracted by Df and so that the quotient $TM|_K/E^s$ is expanded by the action induced by Df .

Then for every $z \in K$, the following set, called *stable manifold* of z , is a $\dim E^s$ -manifold, injectively C^r -immersed into M :

$$W^s(z; f) := \{z' \in M : \lim_{+\infty} d(f^n(z), f^n(z')) = 0\}.$$

The notion of unstable manifold needs to consider the *space of preorbits* $\overleftarrow{K} := \{(z_i)_{i \leq 0} \in K^{\mathbb{Z}^-} : z_{i+1} = f(z_i) \forall i < 0\}$ of K . Given a preorbit $\underline{z} = (z_i)_{i \leq -1} \in \overleftarrow{K}$, we can define the *unstable manifold* of \underline{z} , which is a $\text{codim } E^s$ -manifold C^r -immersed into M :

$$W^u(\underline{z}; f) := \{z' \in M : \exists \underline{z}' \in \overleftarrow{M}, z'_0 = z' \text{ and } \lim_{+\infty} d(z_{-n}, z'_{-n}) = 0\}.$$

In general this manifold is *not* immersed *injectively*.

When $z \in K$ is periodic, the unstable manifold $W^u(z; f)$ denotes the one associated to the unique preorbit of z which is periodic.

A local stable manifold $W_{loc}^s(z; f)$ of z is an embedded, connected submanifold equal to a neighborhood of z in $W^s(z; f)$. The local unstable manifolds are defined similarly. We can chose them so that they depend continuously on z and \underline{z} respectively.

We endow \overleftarrow{K} with the topology induced by the product topology of $K^{\mathbb{Z}}$. Hence \overleftarrow{K} is compact. Note that when $f|_K$ is bijective, \overleftarrow{K} is homeomorphic to K .

We recall that for f' C^1 -close to f , there exists a continuous map $i_{f'} : \overleftarrow{K} \rightarrow M$ which is close to the 0-coordinate projection so that, with \overleftarrow{f} the shift map on \overleftarrow{K} it holds:

$$i_{f'} \circ \overleftarrow{f} = f' \circ i_{f'}.$$

The set $K' := i(f')(\overleftarrow{K})$ is called the *hyperbolic continuation* of \overleftarrow{K} . This conjugacy defines a homeomorphism $\overleftarrow{i}_{f'} = (i_{f'}(\overleftarrow{f}^n(\cdot)))_{n \leq 0}$ between \overleftarrow{K} and the inverse limit \overleftarrow{K}' of K' for f' . Also given $\overleftarrow{z} \in \overleftarrow{K}$ and any local unstable manifold $W_{loc}^u(\overleftarrow{z}; f)$, for f' C^1 -close to f , there is a local unstable manifolds $W_{loc}^u(\overleftarrow{z}; f')$ of $\overleftarrow{i}_{f'}(\overleftarrow{z})$.

1.2.1 Blender

A hyperbolic set K of a surface local diffeomorphism f is a Bonatti-Diaz' *blender* [BD96] if $\dim E^u = 1$ and a continuous union of local unstable manifolds $\cup_{\overleftarrow{z} \in \overleftarrow{K}} W_{loc}^u(\overleftarrow{z}; f)$ contains robustly a non-empty open set O of M :

$$\bigcup_{\overleftarrow{z} \in \overleftarrow{K}} W_{loc}^u(\overleftarrow{z}; f') \supset O \quad , \quad \forall f' \text{ } C^1\text{-close to } f.$$

The set O is called a *covered domain* of the blender K .

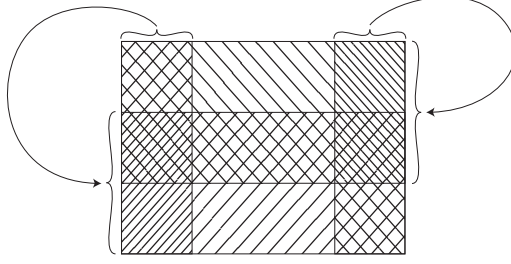


Figure 1: A blender of a surface.

Example 1.5. Let $I_{-1} \sqcup I_{+1}$ be a disjoint union of non-trivial segments in $(-1, 1)$. Let σ be a map which sends affinely each $I_{\pm 1}$ onto $[-1, 1]$. Put:

$$f : (x, y) \in [-2, 2] \times I_{-1} \sqcup I_{+1} \mapsto \begin{cases} f_{+1}(x, y) = (\frac{2}{3}(x-1) + 1, \sigma(y)) & \text{if } y \in I_{+1} \\ f_{-1}(x, y) = (\frac{2}{3}(x+1) - 1, \sigma(y)) & \text{if } y \in I_{-1} \end{cases}$$

Let $K := \cap_{n \geq 0} \sigma^{-n}(I_{-1} \sqcup I_{+1})$ be the maximal invariant of σ , and let $B := [-1, 1] \times K$. We notice that B is a hyperbolic set for f with vertical stable direction and horizontal unstable direction. Given a presequence $\underline{\mathbf{b}} = (\mathbf{b}_i)_{i \leq -1} \in \{-1, 1\}^{\mathbb{Z}^-}$ we define the local unstable manifold:

$$W_{loc}^u(\underline{\mathbf{b}}; f) := \bigcap_{i \geq 1} f^i([-2, 2] \times I_{\mathbf{b}_{-i}}).$$

We notice that for any C^1 -perturbation f' of f , the following is a hyperbolic continuation of $W_{loc}^u(\underline{\mathbf{b}}; f)$:

$$W_{loc}^u(\underline{\mathbf{b}}; f') := \bigcap_{i \geq 1} f'^i([-2, 2] \times I_{\mathbf{b}_{-i}}).$$

Fact 1.6. B is a blender for f , and its covered domain contains $O := (-2/3, 2/3) \times (-1, 1)$.

Proof. Let us notice that B satisfies the following *covering property*. With $O_{+1} := [0, 2/3) \times (-1, 1)$ and $O_{-1} := (-2/3, 0) \times (-1, 1)$, it holds:

$$O = O_{+1} \cup O_{-1}, \quad cl(f_{-1}^{-1}(O_{-1}) \cup f_{+1}^{-1}(O_{+1})) \subset O.$$

Hence, for any perturbation of the dynamics, for any $z \in O$, there exists a preorbit $(z_i)_{i \leq 0}$ so that z_i belongs to $O_{\mathbf{b}_i}$ for $\mathbf{b}_i \in \{\pm 1\}$. With $\underline{\mathbf{b}} = (\mathbf{b}_i)_{i \leq -1}$ we note that $z \in W_{loc}^u(\underline{\mathbf{b}}; f')$. \square

In higher dimension $n \geq 2$, a hyperbolic compact set K with one dimensional unstable direction is a *blender* for a C^1 -dynamics F , if there exists a continuous family of local unstable manifolds $(W_{loc}^u(\overleftarrow{z}; F))_{\overleftarrow{z} \in \overleftarrow{K}}$ whose union intersects robustly a C^1 -neighborhood N of an $n-2$ -dimensional sub-manifold S :

$$\bigcup_{\overleftarrow{z} \in \overleftarrow{K}} W_{loc}^u(\overleftarrow{z}; F') \cap S' \neq \emptyset, \quad \forall S' \in N \quad \forall F' \text{ } C^1 \text{ close to } F.$$

Example 1.7. Let $F := (t, x, y) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R} \mapsto (0, f(x, y))$. We notice that $\{0\} \times B$ is a hyperbolic set of F with one-dimensional unstable direction.

Fact 1.8. *The hyperbolic set $\{0\} \times B$ of F is a blender.*

Proof. We notice that $\mathbb{R}^{n-2} \times \{0\}$ is the strong stable direction. Hence DF^{-1} leaves invariant the constant cone field $\chi = \{(u, v) \in \mathbb{R}^{n-2} \times \mathbb{R}^2 : \|v\| \leq \|u\|\}$.

Let V be the set of C^1 -submanifolds S of the form $S = \text{Graph } \phi$ where $\phi \in C^1((-1, 1)^{n-2}, \mathbb{R}^2)$ so that $\phi(0) \in O$ and $TS \subset S \times \chi$.

We notice that any small C^1 -perturbation F' of F satisfies that DF'^{-1} leaves invariant χ . Hence for every $S \in V$, if $S \cap \{0\} \times \mathbb{R}^2 \subset O_{-1}$ (resp. O_{+1}) then a connected component S' of $F'^{-1}(S) \cap (-1, 1)^{n-2} \times O$ is in V .

Hence, by induction, for every F' C^1 -close to F , for every $S \in V$, we can define a sequence of preimages $(S^i)_i$ associated to symbols $\underline{b} = (b_i)_{i \leq 0}$.

We notice that $(F'^i(S^i))_i$ is a nested sequence of subsets in S , whose intersection $\cap_{i \geq 1} F'^i(S^i)$ consists of a single point z . By shadowing, it comes that z belongs to $W_{loc}^u(\underline{b}; F')$ and so S intersects $\bigcup_{z \in K} W_{loc}^u(z; F')$. \square

1.2.2 λ -blender

In this subsection let us introduce a blender with a special property.

Let M be a manifold, f a self mapping of M which leaves invariant a hyperbolic compact set K with one dimensional unstable direction. Let N_G be an open neighborhood of $E^s|K$ in the Grassmannian bundle GM of TM , which projects onto a neighborhood N of K and satisfies:

$$\forall z \in N \cap f^{-1}(N) \quad D_z f^{-1} N_G(f(z)) \Subset N_G(z), \text{ with } N_G(z) = N_G \cap GM_z$$

Definition 1.9 (λ -Blender when $\dim M = 2$). *The hyperbolic set K is a λ -blender if the following condition is satisfied. There exist a continuous family of local unstable manifolds $(W_{loc}^u(\overleftarrow{z}; f))_{\overleftarrow{z} \in \overleftarrow{K}}$ and a non-empty open set O of $M \times \mathbb{R}$ so that for every f' C^1 -close to f , for every $(Q, \lambda) \in O$, there exists $\overleftarrow{z} \in \overleftarrow{K}$, so that:*

- $Q \in W_{loc}^u(\overleftarrow{z}; f')$, and with $(Q_{-n})_n$ the preorbit of Q associated to \overleftarrow{z} ,
- for every line L in $N_G(Q)$, with $L_n := (D_{Q_{-n}} f^n)^{-1}(L)$, the sequence $(\frac{1}{n} \log \|D_{Q_{-n}} f^n|L_n\|)_n$ converges to λ .

The open set O is called a covered domain for the λ -blender K .

Example 1.10. Let $\mathfrak{B} = \{-1, +1\}^2$ and let $(I_b)_{b \in \mathfrak{B}}$ be four disjoint, non trivial segments in $(-1, 1)$. Let σ be a map which sends affinely each I_b onto $[-1, 1]$. For $\epsilon > 0$ small put:

$$f : (x, y) \in [-2, 2] \times \sqcup_{\mathfrak{B}} I_b \mapsto \left(\left(\frac{2}{3} \right)^{1+\epsilon \delta'} (x - \delta) + \delta, \sigma(y) \right) \quad \text{if } y \in I_b \text{ and } b = (\delta, \delta') .$$

Let $K := \cap_{n \geq 0} \sigma^{-n}(\cup_{b \in \mathfrak{B}} I_b)$ be the maximal invariant of σ , and let $B := [-1, 1] \times K$. We notice that B is a hyperbolic set for f with vertical stable direction and horizontal unstable direction. Let $N_G := \{(u, v) \in \mathbb{R}^2 : \|u\| \geq \|v\|\}$.

Given a presequence $\underline{b} = (b_i)_{i \leq -1}$ and f' C^1 -close to f , we define the local unstable manifold:

$$W_{loc}^u(\underline{b}; f') := \bigcap_{i \geq 1} f'^i([-2, 2] \times I_{b_{-i}}) .$$

Fact 1.11. *The uniformly hyperbolic B is a λ -blender for f , and its covered domain contains:*

$$O := ((-2/3, 2/3) \times (-1, 1)) \times (\log 2/3 - 2\epsilon; \log 2/3 + 2\epsilon) .$$

Proof. Given $\mathbf{b} = (\delta, \delta') \in \{-1, 1\}^2 = \mathfrak{B}$, let

$$O_{\mathbf{b}} = \{((x, y), \lambda) \in O : x \cdot \delta \geq 0, (\lambda - \log 2/3) \cdot \delta' \geq 0\}.$$

Given a C^1 -perturbation f' of the dynamics f , for every $(Q, \lambda) \in O$, given any unit vector $u \in N_G(Q)$ we define inductively a Df' -preorbit $(Q_n, u_n)_n$ associated to a presequence of symbols $(\mathbf{b}_n)_n$ as follows. Put $Q_0 = Q$ and $u_0 = u$. For $n \leq 0$, we define $\mathbf{b}_{n-1} = (\delta_{n-1}, \delta'_{n-1})$ such that δ_{n-1} is the sign of first coordinate of Q_n , and δ'_{n-1} is the sign of $\log \|u_n\| - n\lambda$. By decreasing induction one easily verifies that $(Q_n, \log \|u_n\| - n\lambda + \log 2/3) \in O_{\mathbf{b}_n}$ for every n . Hence $\frac{1}{n} \log \|u_n\| \rightarrow \lambda$ as asked.

As for the proof of Fact 1.11, this implies that $z \in W_{loc}^u(\underline{\mathbf{b}}; f')$, with $\underline{\mathbf{b}} = (\mathbf{b}_{-n})_{n \leq -1}$. \square

Definition 1.12 (λ -blender when $\dim M \geq 3$). *The hyperbolic set K is a λ -parablender if the following condition is satisfied.*

There exist a continuous family of local unstable manifolds $(W_{loc}^u(\overleftarrow{z}; f))_{\overleftarrow{z} \in \overleftarrow{K}}$ and a neighborhood O of a pair (S, λ_0) of a number $\lambda \in \mathbb{R}$ with an $n - 2$ -dimensional C^1 -submanifold S so that, for every $(S', \lambda) \in O$, there exists $\overleftarrow{z} \in \overleftarrow{K}$ satisfying:

- $W_{loc}^u(\overleftarrow{z}; f')$ intersects S' at a point Q , and with $(Q_{-n})_n$ the preorbit of Q associated to \overleftarrow{z} ,
- for $(n-1)$ -plane E in $N_G(Q)$, with $E_n := (D_{Q_{-n}} f^n)^{-1}(E)$, the sequence $(\frac{1}{n} \log \|D_{Q_{-n}} f^n|E\|)_n$ converges to λ .

Example 1.13. Let $F : (t, x, y) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R} \mapsto (0, f(x, y))$ with f as in example 1.10. We notice that $\{0\} \times B$ is a hyperbolic set of F with one-dimensional unstable direction.

In the proof of Fact 1.8, we define a C^1 -open set V of $(n-1)$ -submanifolds.

Fact 1.14. *The hyperbolic set $\{0\} \times B$ of F is a λ -blender with $O = V \times (-\log 2/3 - 2\epsilon, +\log 2/3 + 2\epsilon)$ in its covered domain.*

The proof is done by merging the one of Facts 1.8 and 1.11, and so it is left as an exercise to the reader.

1.3 Hyperbolic theory for families of dynamics

Let us fix $k \geq 0$, $1 \leq r < \infty$, and a C^r -family $\hat{f} = (f_a)_a$ of C^r -maps of M .

1.3.1 Hyperbolic continuation for parameter family

It is well known that if f_0 has a hyperbolic fixed point P_0 , then it persists as a hyperbolic fixed point P_a of f_a for every a small. Moreover, the map $a \mapsto P_a$ is of class C^r .

More generally, if K is a hyperbolic set for f_0 (with possibly $f_0|K$ not injective), it persists in a sense involving its inverse limit \overleftarrow{K} . Let \overleftarrow{f}_0 be the shift map on \overleftarrow{K} .

Theorem 1.15 (Th. 14 [Ber16a]). *For every a in a neighborhood V of 0, there exists a map $h_a \in C^0(\overleftarrow{K}; M)$ so that:*

- h_0 is the zero-coordinate projection $(z_i)_i \mapsto z_0$.
- $f_a \circ h_a = h_a \circ \overleftarrow{f}_0$ for every $a \in V$.
- For every $\underline{z} \in \overleftarrow{K}$, the map $a \in V \mapsto h_a(\underline{z})$ is of class C^r .

The point $h_a(\underline{z})$ is called the *hyperbolic continuation* of \underline{z} for f_a . We denote $\underline{z}_a \in M$ the zero-coordinate of $h_a(\underline{z})$. The family of sets $(K_a)_a$, with $K_a := \{\underline{z}_a : \underline{z} \in \overleftarrow{K}\}$, is called the hyperbolic continuation of K .

The local stable and unstable manifolds $W_{loc}^s(z; f_a)$ and $W_{loc}^u(\underline{z}; f_a)$ are canonically chosen so that they depend continuously on a , z and \underline{z} . They are called the *hyperbolic continuations* of $W_{loc}^s(z; f)$ and $W_{loc}^u(\underline{z}; f)$ for f_a . Let us recall:

Proposition 1.16 (Prop 15 [Ber16a]). *For every $z \in K$, the family $(W_{loc}^s(z; f_a))_{a \in V}$ is of class C^r . For every $\underline{z} \in \overleftarrow{K}$, the family $(W_{loc}^u(\underline{z}; f_a))_{a \in V}$ is of class C^r . Both vary continuously with $z \in K$ and $\underline{z} \in \overleftarrow{K}$.*

1.3.2 Parablender

The bifurcation theory studies the hyperbolic continuation of hyperbolic sets and their local stable and unstable manifolds, to find dynamical properties. Hence we shall study the action of C^r -families $\hat{f} = (f_a)_a$ on C^r -jets.

Given a C^r -family of points $\hat{z} = (z_a)_{a \in \mathbb{R}^k}$, its C^r -jet at $a_0 \in \mathbb{R}^k$ is $J_{a_0}^r \hat{z} = \sum_{j=0}^r \frac{\partial_a^j z_{a_0}}{j!} a^{\otimes j}$. Let $J_{a_0}^r M$ be the space of C^r -jets of k -parameters, C^r -families of points in M .

We notice that any C^r -family $\hat{f} = (f_a)_a$ of C^r -maps f_a of M acts canonically on $J_{a_0}^r M$ as the map:

$$J_{a_0}^r \hat{f}: J_{a_0}^r(z_a)_a \in J_{a_0}^r M \mapsto J_{a_0}^r(f_a(z_a))_a \in J_{a_0}^r M.$$

The first example of parablender was given in [Ber16b]; in [BCP16] a new example of parablender was given. Let us give for the first time a definition for a dynamics on a manifold M of any dimension n .

Definition 1.17 (C^r -Parablender when $\dim M = 2$). *A family $(K_a)_a$ of blenders K_a endowed with a continuous family of local (one dimensional) unstable manifolds $(W_{loc}^u(\underline{z}; f_a))_{\underline{z} \in \overleftarrow{K}}$ is a C^r -parablender at $a = a_0$ if the following condition is satisfied. There exists a non-empty open set O of C^r -families of 2-codimensional C^r -submanifolds so that for every $(f'_a)_a$ C^r -close to $(f_a)_a$, for every $(S_a)_{a \in \mathbb{R}^k} \in O$, there exist $\underline{z} \in \overleftarrow{K}$ and a C^r -curve of points $\hat{Q} = (Q_a)_a$ in $(W_{loc}^u(\underline{z}; f'_a))_a$ and a C^r -curve of points $\hat{P} = (P_a)_a$ in $(S_a)_a$ satisfying:*

$$J_{a_0}^r \hat{Q} = J_{a_0}^r \hat{P}.$$

The open set O is called a covered domain for the C^r -parablender $(K_a)_a$.

Remark 1.18. When $n = 2$, the set O is an open subset of family of points $(S_a)_a \in C^r(\mathbb{R}^k, M)$.

Remark 1.19. We notice that if $J_{a_0}^r(K_a)_a := \{J_{a_0}^r(\underline{z}_a)_a : \underline{z} \in \overleftarrow{K}\}$ is a blender for $J_{a_0}^r(f_a)_a$ then $(K_a)_a$ is a C^r -parablender at a_0 for $(f_a)_a$. We do not know if it is a necessary condition.

Example 1.20 (C^r -Parablender in dimension 2). Let $\Delta_r := \{-1, 1\}^{E_r}$ with $E_r := \{i = (i_1, \dots, i_k) \in \{0, \dots, r\}^k : i_1 + \dots + i_k \leq r\}$. For $\delta \in \Delta_r$ we put:

$$P_\delta(a) = \sum_{i \in E_r} \delta(i) \cdot a_1^{i_1} \cdots a_k^{i_k}.$$

Consider $\text{Card } \Delta_r$ disjoint segments $D_r := \sqcup_{a \in \Delta_r} I_\delta$ of $(-1, 1)$. Let $\sigma: \sqcup_{\delta \in \Delta_r} I_\delta \rightarrow [-1, 1]$ be a locally affine, orientation preserving map which sends each I_δ onto $[-1, 1]$. Let $(f_a)_a$ be the k -parameters family defined by:

$$f_a(x, y): (x, y) \in [-3, 3] \times D_r \mapsto \left(\frac{2}{3}(x - P_\delta(a)) + P_\delta(a), \sigma(y)\right) \quad \text{if } y \in I_\delta.$$

We notice that the maximal invariant set of f_0 is a blender K .

Let us define the following subsets of the space $C_0^r(\mathbb{R}^k, M)$ of germs at 0 of C^r -functions:

$$\hat{O}_r := \{\hat{z} \in C_0^r(\mathbb{R}^k, M) : J_0^r \hat{z} = \sum_{i \in E_r} (x_i, y_i) \cdot a_1^{i_1} \cdots a_k^{i_k} \text{ and } |x_i| < 1, |y_i| < 2/3\}.$$

$$\hat{O}_\delta := \{\hat{z} \in C_0^r(\mathbb{R}^k, M) : J_0^r \hat{z} = \sum_{i \in E_r} (x_i, y_i) \cdot a_1^{i_1} \cdots a_k^{i_k} : |x_i| < 1, 0 \leq \delta(i) \cdot y_i < 2/3\}.$$

We observe that $\hat{O}_r = \cup_{\delta \in \Delta_r} \hat{O}_\delta$. Also for every $\delta \in \Delta_r$, the inverse of $J_0^r(f_a)_a$ maps $cl(\hat{O}_\delta)$ into the interior of \hat{O}_r . Hence by proceeding as in [Ber16a, Example 19], we prove that the hyperbolic continuation $(K_a)_a$ of K is a C^r -parablender at $a_0 = 0$ with \hat{O} included in its covered domain.

Example 1.21 (C^r -Parablender in dimension $n \geq 2$). Let $(f_a)_a$ be given by previous example 1.20 with parablender $(K_a)_a$ and covered domain $\hat{O}_r = \cup_{\delta \in \Delta_r} \hat{O}_\delta$. Let $\hat{F} = (F_a)_a$ be defined by:

$$F_a(x, y) : (t, x, y) \in (-1, 1)^{n-2} \times [-3, 3] \times D_r \mapsto (0, f_a(x, y)).$$

We notice that $(\{0\} \times K_a)_a$ is a family of hyperbolic sets for $(F_a)_a$. Let us show that it is a C^r -parablender.

Let \hat{V}_r be the space of germs at $a = 0$ of C^r -family $\hat{\phi} = (\phi_a)_a$ of C^r -maps $\phi_a \in C^r((-1, 1)^{n-2}, \mathbb{R}^2)$ so that:

- $(\phi_a(0))_a \in \hat{O}_r$,
- the map $(a, t) \mapsto \phi_a(t) - \phi_a(0)$ has C^r -norm smaller than 1.

We identify \hat{V}_r with an open set O of germs at $a = 0$ of C^r -family of C^r -($n-2$)-submanifolds of \mathbb{R}^n by associating to $\hat{\phi}$ its family of graphs $(Graph \phi_a)_a$. Let us show that for every \hat{F}' C^r -close to \hat{F} , for every $(S_a)_a \in O$, there exists $\underline{\delta} \in \Delta^{\mathbb{Z}^-}$, $(P_a)_a \in (S_a)_a$ and $(Q_a)_a \in (W_{loc}^u(\underline{\delta}; f_a))_a$ so that $J_0^r(P_a)_a = J_0^r(Q_a)_a$.

For every $\delta \in \Delta$, we define \hat{V}_δ as the subset of $\hat{\phi} \in \hat{V}_r$ so that $J_0^r(\phi_a(0))_a$ belongs to \hat{O}_δ . Note that $\hat{V}_r = \cup_{\delta \in \Delta} \hat{V}_\delta$. Given any C^r -perturbation \hat{F}' of the family \hat{F} and every $\hat{\phi} \in \hat{V}_\delta$ we define:

$$\hat{\phi}_\delta = (\phi_{a\delta})_a, \quad \text{with } Graph \phi_{a\delta} = (F'_a|_{(-1, 1)^{n-2} \times [-3, 3] \times I_\delta})^{-1} Graph \phi_a \quad \forall a \text{ small.}$$

Fact 1.22. For every \hat{F}' C^r -close to \hat{F} , for every $\hat{\phi} \in \hat{V}_\delta$, the family $\hat{\phi}_\delta$ is well defined and in \hat{V}_r .

Proof. If $\hat{F}' = \hat{F}$, for every $\hat{\phi}$ in \hat{V}_δ , the family $\hat{\phi}_\delta$ is well defined by transversality of the map $(a, t, x, y) \mapsto (a, F_a(t, x, y))$ with the submanifold $\cup_{a \in (-1, 1)^{n-2}} \{a\} \times Graph \phi_a$. Furthermore, the family $\hat{\phi}_\delta$ is equal to $(t \mapsto g_a^\delta \circ \phi_a(0))_{a \in (-1, 1)^{n-2}}$, where g_a^δ is the inverse of $f_a|_{[-3, 3] \times I_\delta}$. As $J_0^d(\phi_a(0))_a$ is in \hat{O}_δ , it comes that $J_0^d(g_a^\delta \circ \phi_a(0))_a$ is in $J_0^d(g_a^\delta)_a(\hat{O}_\delta) \subseteq \hat{O}_r$. Note that $\hat{\phi}_\delta$ is in a subset of \hat{V}_r at positive distance to the complement of \hat{V}_r .

Hence by transversality, for every \hat{F}' in a C^r -small neighborhood of \hat{F} , for every $\hat{\phi} \in \hat{V}_\delta$, the family $\hat{\phi}_\delta$ is in \hat{V}_r . \square

From the latter fact, for every \hat{F}' in a C^r -small neighborhood of \hat{F} , for every $\hat{\phi} \in \hat{V}_r$, we can defined a sequence $\underline{\delta} \in \Delta^{\mathbb{Z}^-}$ and preimages $(\hat{\phi}^n)_{n \leq -1} \in V_r^{\mathbb{Z}^-}$ with $\hat{\phi}^n = \hat{\phi}_{\delta_n^{n+1}}$ and $\hat{\phi}^0 = \hat{\phi}$.

Let $S_a^n = Graph \phi_a^n$ and observe that S_a^n is mapped into $S_a^{n+1} = Graph \phi_a^{n+1}$ for every $n \leq -1$.

Hence the point $P_a^n := (0, \phi_a^n(0))$ is well defined for a small, and by contraction of $F_a^n|_{S_a^n}$, $F_a^m(P_a^n)$ belongs to S_a^0 and the jets $J_0^r(F_a^{n-k}(P_a^n))_a$ are bounded for every $n \geq k \geq 0$.

Hence $(J_0^r F_a^n(P_a^{-n}))_n$ converges to the C^r -jet at $a = 0$ of a C^r -curve of points $(P_a)_a \in (S_a)_a$, and displays a preorbit $((P_k)_a)_{k \leq -1}$ associated to $\underline{\delta}$ with bounded C^r -jet at 0 for every $k \leq -1$.

By the shadowing property of the hyperbolic set $J_0^r(\{0\} \times K_a)_a$ for $J_0^r \hat{F}'$, the point $J_0^r(P_a)_a$ belongs to the unstable manifold of $J_0^r(\{0\} \times K_a)_a$ associated to $\underline{\delta}$. In other words, there exists a C^r -curve $(Q_a)_a \in (W_{loc}^u(\underline{\delta}; F'_a))_a$ so that $J_0^r(Q_a)_a = J_0^r(P_a)_a$.

1.3.3 λ -parablender

Let us generalize the notion of λ -blender to its parametric version the λ -parablender for a C^r -family of dynamics $(f_a)_a$ of a manifold M of dimension n .

For this end, given $(\hat{z}, \hat{u}) \in C^{r-1}(\mathbb{R}^k, TM)$ so that $\hat{z} \in C^r(\mathbb{R}^k, M)$, we consider:

$$\check{J}_{a_0}^r(\hat{z}, \hat{u}) := (J_{a_0}^r \hat{z}, J_{a_0}^{r-1} \hat{u}) \quad \text{and} \quad \check{J}_{a_0}^r TM := \{\check{J}_{a_0}^r(\hat{z}, \hat{u}) : (\hat{z}, \hat{u}) \in C^r(\mathbb{R}^k, TM)\}$$

We note that $D\hat{f}$ acts canonically on $\check{J}_{a_0}^r TM$ as:

$$\check{J}_{a_0}^r D\hat{f} \circ \check{J}_{a_0}^r(\hat{z}, \hat{u}) = \check{J}_{a_0}^r(\hat{f} \circ \hat{z}, D_{\hat{z}} \hat{f}(\hat{u}))$$

Let $(K_a)_a$ be a family of C^r -parablenders for a C^r -family of dynamics $(f_a)_a$. Let N_G be a neighborhood of the stable direction of K_0 in the Grassmanian bundle GM of TM , which projects onto a neighborhood N of K_0 and satisfies:

$$\forall z \in N \cap f_0^{-1}(N) \quad D_z f_0^{-1} N_G(f_0(z)) \subseteq N_G(z).$$

Let $\check{J}_0^r N_G \subset \check{J}_0^r TM$ be the subset of jets $(J_0^r(z_a)_a, J_0^{r-1}(u_a)_a) \in \check{J}_0^r TM \setminus \{0\}$ so that $u_a \in N_G(z_a)$ for every a .

Let $\hat{\mathcal{V}}$ be the space of C^r -families $\hat{S} = (S_a)_a$ of 2-codimensional submanifolds S_a . We notice that $\hat{\mathcal{V}}$ is equal to $C^r(\mathbb{R}^k, M)$ if $n = 2$.

Definition 1.23 (C^r - λ -Parablender). *A family $(K_a)_a$ of blenders for $(f_a)_a$ is a C^r - λ -parablender at $a = a_0$ if the following condition is satisfied.*

There exists a continuous family of local unstable manifolds $(W_{loc}^u(\overleftarrow{z}; f_a))_{\overleftarrow{z} \in \overleftarrow{K}}$, a non-empty open set O of $\hat{\mathcal{V}} \times C^{r-1}(\mathbb{R}^k, (-\infty, 0))$ so that for every $(f'_a)_a$ C^r -close to $(f_a)_a$, for every $(\hat{S}, \hat{\lambda}) \in O$, there exist $\overleftarrow{z} \in \overleftarrow{K}$ and a C^r -curve of points $\hat{Q} = (Q_a)_a \in (W_{loc}^u(\overleftarrow{z}; f'_a))_a$ and $\hat{P} = (P_a)_a \in (S_a)_a$ satisfying:

$$J_{a_0}^r \hat{P} = J_{a_0}^r \hat{Q}.$$

Furthermore, for every C^{r-1} -family of $(n-1)$ -planes $(E_a)_a$ in $(N_G(Q_a))_a$ and $(Q_{-n\ a}, E_{-n\ a})_a$ the preimage by $(Df_a^n)_a$ of $(Q_a, E_a)_a$ associated to the preorbit \overleftarrow{z} , it holds:

$$\lim_{n \rightarrow \infty} \frac{1}{n} J_{a_0}^{r-1}(\log \|D_{Q_{-n\ a}} f_a^n|_{E_{-n\ a}}\|)_a = J_{a_0}^{r-1} \hat{\lambda}.$$

The open set O is called a covered domain for the C^r - λ -parablender $(K_a)_a$.

Remark 1.24. In particular $(K_a)_a$ is a C^r -parablender and K_0 is a λ -blender.

Example 1.25 (C^r - λ -Parablender in dimension 2). Let E_r , Δ_r and P_δ be defined as in Example 1.20 and put $\mathfrak{B} := \Delta_r \times \Delta_{r-1}$. Let \hat{O}_r , \hat{O}_δ be the subset of $C^r(\mathbb{R}^k, M)$ defined therein.

Consider $\text{Card}(\mathfrak{B})$ disjoint segments $D := \sqcup_{\mathfrak{a} \in \mathfrak{B}} I_{\mathfrak{a}}$ of $(-1, 1)$. Let $\sigma : \sqcup_{\mathfrak{a} \in \mathfrak{B}} I_{\mathfrak{a}} \rightarrow [-1, 1]$ be a locally affine map which sends each $I_{\mathfrak{a}}$ onto $[-1, 1]$. For $\epsilon > 0$ small, let $(\tilde{f}_a)_a$ be the k -parameters family defined by:

$$\tilde{f}_a : (x, y) \in D \times [-3, 3] \mapsto \left(\frac{2}{3} \cdot \exp(\epsilon \cdot P_{\delta'}(a)) \cdot x + \frac{P_\delta(a)}{3}, \sigma(y)\right) \quad \text{if } y \in I_{\mathfrak{a}}, \mathfrak{a} = (\delta, \delta').$$

We notice that the maximal invariant set \tilde{K}_a of \tilde{f}_a in $[-3, 3] \times D$ is hyperbolic and for $\underline{\mathbf{b}} = (\mathbf{b}_i)_{i \leq -1} \in \mathfrak{B}^{\mathbb{Z}^-}$, with local unstable manifold $W_{loc}^u(\underline{\mathbf{b}}, \tilde{f}_a) := \bigcap_{i \geq 1} \tilde{f}_a^i([-2, 2] \times I_{\mathbf{b}_{-i}})$.

Let N_G be the cone field constantly equal $\{(u, v) : \|u\| \leq \|v\|\}$. We notice that it is backward invariant. Let:

$$\tilde{O} = \hat{O}_r \times \{\hat{\lambda} \in C_0^{r-1}(\mathbb{R}^k, \mathbb{R}) : J_0^{r-1} \hat{\lambda} = \log \frac{2}{3} + \sum_{i \in E_{r-1}} \lambda_i a^i : \lambda_i \in [-2\epsilon, 2\epsilon]\}.$$

Fact 1.26. $(\tilde{K}_a)_a$ is a λ - C^r -parablender for $(\tilde{f}_a)_a$, and its covered domain contains \tilde{O} .

Proof. We consider the covering $(\tilde{O}_{\mathbf{b}})_{\mathbf{b} \in \mathfrak{B}}$ of \tilde{O} with for every $\mathbf{b} = (\delta, \delta') \in \mathfrak{B}$:

$$\tilde{O}_{\mathbf{b}} := \hat{O}_{\delta} \times \{\hat{\lambda} \in C_0^{r-1}(\mathbb{R}^k, \mathbb{R}) : J_0^{r-1} \hat{\lambda} = \log \frac{2}{3} + \sum_{i \in E_{r-1}} \lambda_i a^i : \delta'_i \cdot \lambda_i \in [0, 2\epsilon]\},$$

and proceed by merging the proofs of Fact 1.11 and Example 1.20. \square

Example 1.27 (C^r - λ -Parablender in dimension $n \geq 2$). Let $(\tilde{f}_a)_a$ be given by previous example 1.25 with λ -parablender $(\tilde{K}_a)_a$ and covered domain $\tilde{O}_r = \bigcup_{\mathbf{b} \in \mathfrak{B}} \tilde{O}_{\mathbf{b}}$. Let $(\tilde{F}_a)_a$ be defined by:

$$\tilde{F}_a(x, y) : (t, x, y) \in (-1, 1)^{n-2} \times [-3, 3] \times D \mapsto (0, \tilde{f}_a(x, y)).$$

Let $\tilde{B}_a := \{0\} \times \tilde{K}_a$ and note that $(\tilde{B}_a)_a$ is a family of hyperbolic sets for $(F_a)_a$. Let \hat{V}_r be the subspace of C^r -families of $(n-2)$ -dimensional submanifolds defined in Example 1.21.

Fact 1.28. The family of hyperbolic sets $(\tilde{B}_a)_a$ for $(\tilde{F}_a)_a$ is a λ - C^r -parablender with covered domain:

$$\hat{V}_r \times \{\hat{\lambda} : J_0^{r-1} \hat{\lambda} = \log \frac{2}{3} + \sum_{i \in E_{r-1}} \lambda_i a^i : \lambda_i \in [-2\epsilon, 2\epsilon]\}.$$

Proof. Similarly to Fact 1.26, the proof is done by merging those of Fact 1.14 and Example 1.21. \square

Remark 1.29. A possible alternative prove of the main theorem would be to generalize the concept of λ - C^r -parablender in order to obtain not only a control on the parameter jets of points and of the first differential, but also the r -first derivatives. In this manuscript we prefer to exhibit diophantine rotations since they can be useful for many other purposes. Also the necessary condition describing the open set of families should be easier to exhibit typically (to solve conjecture 0.6).

1.4 Normally hyperbolic fibrations

Let M be a manifold and $r \geq 1$. Let K be a compact set and let $(N_y)_{y \in K}$ be a continuous family of disjoint, compact C^r -submanifolds of M .

Let $\mathcal{L} = \bigcup_{y \in K} N_y \rightarrow K$ be the compact bundle with fibers $(N_y)_{y \in K}$.

A map $f \in C^r(M, M)$ leaves invariant \mathcal{L} if for every $y \in K$, there exists $\sigma(y) \in K$ so that f sends N_y into $N_{\sigma(y)}$.

Then observe that Df leaves invariant the tangent bundle to the fibers $T\mathcal{L} := \bigcup_{y \in K} TN_y$. Hence the action $[Df]$ of Df on the normal bundle $\mathcal{N} = TM|_{\mathcal{L}}/T\mathcal{L}$ is well defined.

For $\rho \geq 1$, the dynamics is ρ -normally hyperbolic at \mathcal{L} if there exists a splitting $E^s \oplus E^u = \mathcal{N}$ so that:

- $[Df]$ leaves invariant E^s and E^u :

$$[Df](E^s) = E^s \quad \text{and} \quad [Df](E^u) = E^u ,$$

- for all unit vectors $v_s \in E^s$, $v_u \in E^u$, and $v \in T\mathcal{L}$, it holds:

$$\|[Df(v_s)]\| < 1 \quad , \quad \|[Df(v_u)]\| > 1 \quad \text{and} \quad \|[Df(v)]\| > \|Df(v)\|^\rho > \|[Df(v_s)]\| .$$

The dynamics is ρ -normally expanding at \mathcal{L} if $E^s = 0$.

Normally hyperbolic fibrations are important since they are persistent:

Theorem 1.30 ([HPS77], [Ber10]). *Let $r \geq 1$ and $1 \leq \rho \leq r$, with $\rho < \infty$. Let f be a C^r -map of M which is ρ -normally hyperbolic at the bundle $\mathcal{L} = \cup_{y \in K} N_y$. Moreover, if f is not a diffeomorphism, we assume that f is ρ -normally expanding at \mathcal{L} . Then, for any C^r -perturbation f' of f , there exists a continuous family $(N'_y)_{y \in K}$ of disjoint C^r -submanifolds so that:*

- $f'(N'_y) = N'_{\sigma(y)}$ for every $y \in K$.
- N'_y is C^r -close to N_y for every $y \in K$.

2 Plan of the proofs

Let $2 \leq r \leq \infty$, let M be a manifold and let f be a C^r -differentiable map of M , which is invertible if $\dim M \geq 3$. The proofs of Theorems A and B are done by considering a continuous family $(\mathbb{T}_y)_{y \in K}$ of disjoint, C^r -embedded circles $\mathbb{T}_y \subset M$, indexed by a Cantor set K . We assume the circle bundle $\mathcal{L} = \cup_{y \in K} \mathbb{T}_y$ left invariant by f and normally hyperbolic.

For our purpose we will choose K so that the induced dynamics is conjugated to a full shift σ on a finite alphabet \mathfrak{A} . Then for a dense set of $y \in K$, the fiber \mathbb{T}_y is p -periodic: $f^p(\mathbb{T}_y) = \mathbb{T}_y$ for some $p \geq 1$.

Proof of Theorem A The following proposition provides a sufficient condition to prove Theorem A:

Proposition 2.1. *Let $2 \leq r \leq \infty$. Assume the existence of a non-empty set U of $C^r(M, M)$ and a dense set $D \subset U \cap C^\infty(M, M)$ so that for every $f \in D$ the following property holds.*

There exists a p -periodic, normally hyperbolic, C^∞ -circle \mathbb{T}_y , so that $f^p|_{\mathbb{T}_y}$ is C^∞ -conjugated to an irrational rotation.

Then given any $(a_n)_n \in \mathbb{N}^\mathbb{N}$, a topologically generic $f \in U$ satisfies (\star) .

Proof. As p is the period of \mathbb{T}_y , we notice that $f^j(\mathbb{T}_y)$ disjoint to \mathbb{T}_y for every $1 \leq j < p$. Thus, for a small C^∞ -perturbation f' of f , the map f'^p leaves invariant \mathbb{T}_y and $f'^p|_{\mathbb{T}_y}$ is conjugated to a rational rotation. Then there exists a $q \in p\mathbb{N} \setminus \{0\}$ minimal so that f'^q leaves invariant \mathbb{T}_y , and $f'^q|_{\mathbb{T}_y} = id_{\mathbb{T}_y}$. We notice that q is large when f' is close to f .

Also there exist non trivial open intervals $J \Subset I \subset \mathbb{T}_y$ so that $(f'^i(I))_{1 \leq i \leq q}$ is a disjoint family of intervals. Let $\rho \in C^\infty(\mathbb{T}, [0, 1])$ be a function supported by I and so that $\rho|_J = 1$.

We handle a small perturbation f'' of f' supported by a small neighborhood of I – and so disjoint to $\cup_{1 \leq k < q} f'^k(I)$ – so that $f''^q|_J = x \in J \mapsto x + \epsilon \rho(x) \sin(2\pi \cdot a_q x / |J|)$ for $\epsilon > 0$ small.

Note that f'' displays at least a_q periodic points of period q . By normal hyperbolicity, these periodic points are hyperbolic and so persist for small perturbations of f'' .

This proves for every $N \geq 1$, the existence of an open and dense set $U_N \subset U$ formed by maps $f \in U_N$ displaying at least a_q saddle periodic points of period q for a certain $q \geq N$. The topologically generic set is $\mathcal{R} := \bigcap_{N \geq 0} U_N$. We notice that for every $f \in \mathcal{R}$, there exists q arbitrarily large so that $\text{Card } \text{Per}_q^0 f \geq a_q$. \square

Although by KAM theory, it sounds very intuitive that for many normally hyperbolic circle bundles, the hypothesis of the above proposition holds true, it is not obvious: an open and dense set of C^∞ -maps of the circle are Morse-Smale. As the set of periodic fibers in a normally hyperbolic, circle bundle is countable, the dynamics which satisfies the hypothesis of the above proposition is topologically meager!

This is somehow one of the reasons leading Kolmogorov to propose a new concept of typicality (latter formalized by Arnold): he wanted a definition for which an irrational rotation would be typical among surface flows.

To exhibit such an abundance, a key new idea is to focus on dynamics which are parabolic on a periodic embedded circle.

Theorem 2.2. *For every $\infty \geq r \geq 2$, there exists an open set U of C^r -maps f , each of which is normally hyperbolic at a circle bundle $(\mathbb{T}_y)_{y \in K}$ (depending on f), and there exists a C^r -dense set $D \subset U \cap C^\infty(M, M)$ so that the following property holds.*

For every $f \in D$, there exists p -periodic circle \mathbb{T}_y , so that $f^p|_{\mathbb{T}_y}$ is parabolic. Moreover f is a diffeomorphism if $\dim M \geq 3$.

The proof of this theorem is not trivial and will be the subject of next section 3. It involves a normally hyperbolic fibration by circles which is split into two related parts : one which involves a South-North dynamics on the fibers and another one which involves a North-South dynamics forming an attracting blender at the neighborhood of the south pole. The parabolic map is obtained thanks to a renormalization process.

Theorem 2.2 implies Theorem A. Let $f \in D \subset U \cap C^\infty(M, M)$ which is parabolic at a p -periodic, normally hyperbolic circle \mathbb{T}_y . For every $r' \geq 3$, the dynamics f^p is actually r' -normally hyperbolic at \mathbb{T}_y (for an adapted metric). As f is of class C^∞ , the periodic circle is actually of regularity $C^{r'}$ by [HPS77, §Forced smoothness], and so of class C^∞ .

Then Proposition 1.3 implies the existence of a C^∞ -perturbation so that the dynamics on the normally hyperbolic, periodic, C^∞ -circle displays a Diophantine rotation number.

Therefore, by Yoccoz' Theorem 1.1, the dynamics on the normally hyperbolic, periodic, C^∞ -circle is (C^∞ -conjugated to) an irrational rotation. Finally Proposition 2.1 implies immediately Theorem A. \square

Proof of Theorem B The proof of Theorem B follows the same strategy as the one of Theorem A. Let us fix $\infty > r \geq 2$.

This time we are going to consider a circle bundle $(\mathbb{T}_y)_{y \in K}$ which is normally hyperbolic (resp. expanded if $\dim M = 2$) for a map $f \in C^r(M, M)$. By Theorem 1.30, there exists a C^r -neighborhood U of f such that for every $f' \in U$, the bundle persists as $(\mathbb{T}_y(f'))_{y \in K}$ which is f' -invariant and normally hyperbolic. Theorem 2.2 is generalised to the following:

Theorem 2.3. *There exists an open set \hat{U} of C^r -families $(f_a)_a$ of maps f_a which are normally hyperbolic at a circle bundle $(\mathbb{T}_y(f_a))_{y \in K}$, and a C^r -dense set $\hat{D} \subset \hat{U}$ formed by C^∞ -families $(f_a)_a$ satisfying the following property:*

There exists an open covering $(U_i)_i$ of $[-1, 1]^k$, and for every i , there exist a p_i -periodic base point y_i so that $f_a^{p_i}|_{\mathbb{T}_{y_i}(f_a)}$ is parabolic for every $a \in U_i$.

Moreover \hat{U} consists of families of diffeomorphisms if $\dim M \geq 3$.

The proof of this theorem is not trivial and will be the subject to section 4. The proof will be similar to the one of Theorem 2.2, although the blender will be replaced by the λ - C^d -parablender.

Let us notice that Theorems 1.4 and 2.3 imply:

Fact 2.4. *Under the statement of Theorem 2.3, there exists a C^r -dense set $\hat{D}' \subset \hat{U}$ of infinitely smooth families $(f_a)_a \in \hat{D}'$ so that the following property holds.*

There exists an open covering $(U_i)_i$ of $[-1, 1]^k$ so that for every i , there exists a p_i -periodic base point y_i so that $(\mathbb{T}_{y_i}(f_a))_{a \in U_i}$ is of class C^∞ and $f_a^{p_i}|_{\mathbb{T}_{y_i}(f_a)}$ has rotation number constantly equal to a small diophantine number α_i for every $a \in U_i$.

Proof. Theorem 2.3 provides a dense set $\hat{D} \subset \hat{U}$ of C^∞ -families $(f_a)_a$ endowed with a covering $(U_i)_{i=1}^N$ and a family $(y_i)_i \in K^N$ of periodic points of period $(p_i)_{i=1}^N \in \mathbb{N}^N$. For every $r' \geq 1$, there exists an adapted metric so that the map $(a, z) \mapsto (a, f_a^{p_i}(z))$ is r' -normally hyperbolic at $\cup_{a \in U_i} \{a\} \times \mathbb{T}_{y_i}(f_a)$. Hence by the family $(\mathbb{T}_{y_i}(f_a))_{a \in U_i}$ is of class $C^{r'}$ by [HPS77, §Forced smoothness], and so of class C^∞ .

Up to replace U_i by $U_i \cup U_j$, we can assume the orbits of y_i and y_j disjoint for $i \neq j$.

Then the distance $d(\mathbb{T}_{y_i}(f_a), \mathbb{T}_{y_j}(f_a))$ is bounded from below for every $a \in U_i \cap U_j$. Hence for a subcovering $(U'_i)_i$ of $(U_i)_i$, we can perturb independently each $(f_a^{p_i}|_{\mathbb{T}_{y_i}(f_a)})_{a \in U'_i}$, among $1 \leq i \leq N$. Then Theorem 1.4 accomplishes the proof of this fact. \square

Then by the second part of Herman-Yoccoz' Theorem 1.1, the family $f_a^{p_i}|_{\mathbb{T}_{y_i}(f_a)}$ is C^∞ -conjugated to the rotation R_{α_i} for every $a \in U_i$, and the conjugacy $h_{i,a}$ depends infinitely smoothly on a . Hence by the same reasoning as for the proof of Proposition 2.1, there exist $q_i \in \mathbb{N} \setminus \{0\}$ arbitrarily large and a C^∞ -perturbation $(f'_a)_{a \in U_i}$ of $(f_a)_{a \in U_i}$ so that $f_a^{q_i \cdot p_i}$ leaves invariant $\mathbb{T}_{y_i}(f_a)$ and $f_a^{q_i \cdot p_i}|_{\mathbb{T}_{y_i}(f_a)} = id_{\mathbb{T}_{y_i}(f_a)}$ for every $a \in U_i$. This proves:

Fact 2.5. *Under the statement of Theorem 2.3, for every $N \geq 1$, there exists a dense set $\hat{D}'' \subset \hat{U}$ of C^∞ -families $(f_a)_a$ so that the following property holds.*

There exists an open covering $(U_i)_i$ of $[-1, 1]^k$, such that for every i , there exist a p_i -periodic point $y_i \in K$ and $q_i \geq N$ so that $f_a^{p_i \cdot q_i}|_{\mathbb{T}_{y_i}(f_a)}$ is the identity for every $a \in U_i$, and $f_a^{p_i \cdot q}|_{\mathbb{T}_{y_i}(f_a)} \neq id$ for every $0 < q < q_i$.

Proof of Theorem B. The latter Fact gives for every $(f_a)_a \in \hat{D}'$, a finite covering $(U_i)_i$ of $[-1, 1]^k$ associated to $p_i, q_i \geq 1$ and $y_i \in K$. Let $(U'_i)_i$ be an open covering of $[-1, 1]^k$ so that $cl(U'_i) \subset U_i$ for every i .

From the proof of fact 2.4, we can assume the orbits of y_i and y_j disjoint for $i \neq j$. Furthermore, for every i , there exists a non trivial, open interval $J_a \subset \mathbb{T}_{y_i}(f_a)$ so that the family $(f_a^k(J_a))_{0 \leq k < q_i \cdot p_i}$ is formed by disjoint segments for every $a \in U_i$. We notice that we can chose J_a depending smoothly on a . By a coordinate change, we can assume that $J_a = J$ does not depend on a .

Let $I \Subset J$ be a non trivial segment and let $\rho_i \in C^\infty(U_i \times \mathbb{T}, [0, 1])$ be a function with support in $U_i \times J$ and such that $\rho|_{U'_i \times I} = 1$.

As in the proof of Proposition 2.1, we handle a small perturbation f'_a of f_a supported by a small neighborhood of $\mathbb{T}_{y_i}(f_a)$ (and so disjoint to the union of $\cup_{1 \leq k < q_i \cdot p_i} f^k(\mathbb{T}_{y_i}(f_a))$ with the orbit of the others $\mathbb{T}_{y_j}(f_a)$ for $j \neq i$) so that for $\epsilon > 0$ small, for every $a \in U_i$ and i :

- f'_a sends $J_a \subset \mathbb{T}_{y_i}(f_a)$ into itself,
- f'_a sends $J_a = x \mapsto x + \epsilon \rho(a, x) \sin(2\pi a_{p_i \cdot q_i} x / |J|) \in \mathbb{T}_{y_i}(f_a)$, with $(a_n)_n$ given by Theorem B assumptions.

Then for every i , the map f'_a displays at least $a_{p_i \cdot q_i}$ saddle periodic points of period $p_i \cdot q_i$ for every $a \in U_i$. As these periodic points persist for small perturbations of f'_a , this proves the existence of an open and dense set $\hat{U}_N \subset \hat{U}$ so that for every $(f_a)_a \in \hat{U}_N$, for every $a \in [-1, 1]^k$, the map f_a displays at least $a_{p_i \cdot q_i}$ saddle points of period $p_i \cdot q_i \geq N$.

Note that the intersection $\hat{\mathcal{R}} := \cap_{N \geq 0} \hat{U}_N$ is C^r -topologically generic and for every $(f_a)_a \in \hat{\mathcal{R}}$, for every $a \in [-1, 1]^k$, there exists p arbitrarily large so that $\text{Card Per}_p^0 f_a \geq a_p$. \square

3 Density of parabolic circle map in normally hyperbolic circle bundles

We are going to prove Theorem 2.2 for surface self-mappings and then we will extend the construction for diffeomorphisms in higher dimensions.

Let \mathfrak{c} be a symbol and let $\mathfrak{B} = \{-1, +1\}^2$. We define the finite alphabet $\mathfrak{A} = \mathfrak{B} \cup \{\mathfrak{c}\}$.

Let $\sqcup_{\mathfrak{a} \in \mathfrak{A}} I_{\mathfrak{a}}$ be a disjoint union of non trivial intervals in $(-1, 1)$ with length $< 2/3$.

Let us consider a map $\sigma: [-2, 2] \rightarrow [-2, 2]$ so that σ sends affinely each $I_{\mathfrak{a}}$ onto $[-1, 1]$ and so that σ is equal to the identity nearby $\{-2, 2\}$. We notice that $|D_x \sigma| > 3$, $\forall x \in \sqcup_{\mathfrak{a}} I_{\mathfrak{a}}$.

The circle \mathbb{T} is the one point compactification of \mathbb{R} in the Riemannian sphere. In this coordinate, for every $(a, b, c) \in \mathbb{R}^3$, the map $h_{a,b,c}: x \in \mathbb{T} \mapsto \frac{ax+b}{cx+1} \in \mathbb{T}$ is analytic and invertible if $a > bc$. Let $\epsilon > 0$ be small and put:

$$f_{\mathfrak{c}} = \frac{3}{2} \cdot \frac{x}{x+1}, \quad \text{and} \quad f_{\mathfrak{b}}(x) = (2/3)^{1+\epsilon \cdot \delta'}(x - \delta) + \delta \quad \text{if } \mathfrak{b} = (\delta, \delta') \in \{-1, +1\}^2 = \mathfrak{B}.$$

Let \mathbb{A} be the cylinder $\mathbb{T} \times [-2, 2]$. Let $(f_y)_{y \in [-2, 2]}$ be a C^∞ -family of maps of \mathbb{T} such that:

- f_y is the identity for y close to $\{-2, 2\}$.
- it holds $f_y = f_{\mathfrak{a}}$ if $y \in I_{\mathfrak{a}}$ for $\mathfrak{a} \in \mathfrak{A}$.
- $\|\partial_x f_y\|$ is at most $(3/2)^{1+\epsilon}$ for every $y \in [-2, 2]$.

We are going to show that a locally dense set of perturbations of the following map display a fast growth of the number of periodic points.

$$f: (x, y) \in \mathbb{A} \mapsto (f_y(x), \sigma(y)) \in \mathbb{A}$$

We notice that f is equal to the identity on a neighborhood of $\mathbb{T} \times \{-2, 2\}$.

Fact 3.1. *We can embed the dynamics f into any surface M .*

Proof. For any surface S , we can chose an inclusion $\mathbb{A} \hookrightarrow S$, and endow the surface with the dynamics equal to f on \mathbb{A} and the identity elsewhere. \square

Hence it suffices to focus on perturbations of $f|_{\mathbb{A}}$ to show Theorem 2.2 for any surface.

3.1 Normally expanded circle bundle

The compact set $K := \cap_{n \geq 0} \sigma^{-n}([-1, 1])$ is left invariant by σ and is more than 3-expanded.

We identify $\sigma|K$ to the shift dynamics of $\mathfrak{A}^{\mathbb{N}}$ via the conjugacy:

$$h: \bar{\mathbf{a}} = (\mathbf{a}_i)_{i \geq 0} \in \mathfrak{A}^{\mathbb{N}} \mapsto h(\bar{\mathbf{a}}) \in K \text{ equals to the unique point of } \bigcap_{n \geq 0} \sigma^{-n}(I_{\mathbf{a}_n}) \in K.$$

For every $\bar{\mathbf{a}} \in \mathfrak{A}^{\mathbb{N}}$, we notice that the circle $\mathbb{T}_{\bar{\mathbf{a}}} := \mathbb{T} \times \{h(\bar{\mathbf{a}})\}$ is sent by f onto $\mathbb{T}_{\sigma(\bar{\mathbf{a}})}$. In other words the map f leaves invariant the circle bundle $\mathcal{L} := \cup_{\bar{\mathbf{a}} \in \mathfrak{A}^{\mathbb{N}}} \mathbb{T}_{\bar{\mathbf{a}}}$.

Furthermore, for every $x \in \mathbb{T}$ and $y \in K$, it holds:

$$D_{(x,y)}f(0,1) = (0,1) \cdot D_y\sigma, \quad \text{with } |D_y\sigma| > 3,$$

$$D_{(x,y)}f(1,0) = (1,0) \cdot D_x f_y(x), \quad \text{with } |D_x f_y| < 3/2.$$

Hence the fibration \mathcal{L} is 2-normally expanded and by Theorem 1.30, for every f' C^2 -close to f , there exists a continuous family $(\mathbb{T}'_{\bar{\mathbf{a}}})_{\bar{\mathbf{a}} \in \mathfrak{A}^{\mathbb{N}}}$ of disjoint C^2 -circles which is f' -invariant.

3.2 Basic properties of the dynamics f

South-North dynamics on $\mathbb{T}_{\mathbf{c}^{\mathbb{N}}}$ Let $y_S \in K$ be the unique fixed point of $\sigma|I_{\mathbf{c}}$. In the identification $K \approx \mathfrak{A}^{\mathbb{N}}$, the point y_S is identified to the sequence $\mathbf{c}^{\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$ constantly equal to \mathbf{c} .

We notice that $S := (0, y_S)$ is a fixed point for f which is a projectively hyperbolic source: its eigenvalues are of different moduli. More precisely, the horizontal is the eigenspace associated to the weak unstable eigenvalue $3/2$, whereas the vertical is the eigenspace associated to the strong unstable eigenvalue λ_{uu} of modulus > 3 .

In particular the set $W_{loc}^{uu}(S; f) := \{0\} \times [-1, 1]$ is a *strong unstable manifold* of f : for any $z \in W_{loc}^{uu}(S; f)$, the distance from $(f|_{\mathbb{T} \times I_{\mathbf{c}}})^{-n}(z)$ to S is exponentially small with factor $1/|\lambda_{uu}|$.

On the other hand, the point $N = (1/2, y_S)$ is a saddle point for f . The dynamics $f|_{\mathbb{T}_{\mathbf{c}^{\mathbb{N}}}}$ is South-North, with fixed points S and N .

An important property, for the sequel, is that $f_{\mathbf{c}}$ is conjugated to $D: x \in \mathbb{T} \mapsto 3x/2$ via the parabolic map $h = x/(2x+1)$:

$$f_{\mathbf{c}} = h \circ D \circ h^{-1}.$$

North-South dynamics on $(\mathbb{T}_{\bar{\mathbf{b}}})_{\bar{\mathbf{b}} \in \mathfrak{B}^{\mathbb{N}}}$ In contrast with the latter property of $f_{\mathbf{c}}$, the map $f_{\bar{\mathbf{b}}}$ is conjugated to $(2/3)^{1+\delta\epsilon} \cdot x$ via a translation, for every $\bar{\mathbf{b}} = (\delta, \delta') \in \mathfrak{B}$. Hence the infinity is fixed and expanded by $f_{\bar{\mathbf{b}}}$.

For ϵ -small enough, by Example 1.10, with $K := \cap_{n \geq 0} \sigma^{-n}(I_{\mathbf{b}_-} \sqcup I_{\mathbf{b}_+})$, the set $B = [-1, 1] \times K$ is a λ -blender for f . We recall that for every f' C^1 -close to f , for every $\underline{\mathbf{b}} \in \mathfrak{B}^{\mathbb{Z}^-}$ we define:

$$W_{loc}^u(\underline{\mathbf{b}}; f') := \bigcap_{i \geq 1} f'^i([-2, 2] \times I_{\mathbf{b}_{-i}}), \quad \text{if } \underline{\mathbf{b}} = (\mathbf{b}_i)_{i \leq -1}.$$

Moreover, by Fact 1.11, for $\epsilon > 0$ small enough, it holds:

Fact 3.2. *For every f' C^1 -close to f , there exists $\underline{\mathbf{b}} \in \mathfrak{B}^{\mathbb{Z}^-}$ so that $W_{loc}^u(\underline{\mathbf{b}}; f')$ contains the hyperbolic continuation S' of S , and with S'_{-n} the preimage of S' by f'^n in $\mathbb{T}'_{\mathbf{b}_{-n} \cdots \mathbf{b}_{-1}, \mathbf{c}^{\mathbb{N}}}$, it holds:*

$$\frac{1}{n} \log \|Df'^n|_{T_{S'_{-n}} \mathbb{T}'_{\mathbf{b}_{-n} \cdots \mathbf{b}_{-1}, \mathbf{c}^{\mathbb{N}}}}\| \rightarrow -\log \|Df'|_{T_{S'} \mathbb{T}'_{\mathbf{c}^{\mathbb{N}}}}\|,$$

with $\mathbf{b}_{-n} \cdots \mathbf{b}_{-1} \cdot \mathbf{c}^{\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$ the sequence with n first terms $\mathbf{b}_{-n} \cdots \mathbf{b}_{-1}$ and eventually equals to \mathbf{c} .

3.3 Creation of a semi-parabolic point in \mathcal{L}'

Let f' in a C^1 -neighborhood of f and of class C^∞ .

Let $\underline{b} \in \mathfrak{B}^{\mathbb{Z}^-}$ be given by the Fact 3.2. Let $\underline{b}^{-k} = (\cdots b_{-k-2} \cdot b_{-k-1})$ for every $k \geq 0$.

For every n large, we define the $2n$ -periodic sequence $\bar{\mathbf{p}}^0 = (\mathbf{c} \cdots \mathbf{c} \cdot \mathbf{b}_{-n} \cdots \mathbf{b}_{-1})^{\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$. For every $k \in \mathbb{Z}$, let $\bar{\mathbf{p}}^k$ be the $2n$ -periodic sequence in $\sigma^k(\bar{\mathbf{p}}^0)$. We notice that $\bar{\mathbf{p}}^{-n} = \bar{\mathbf{p}}^n = (\mathbf{b}_{-n} \cdots \mathbf{b}_{-1} \cdot \mathbf{c} \cdots \mathbf{c})^{\mathbb{N}}$ and $\bar{\mathbf{p}}^{-1} = (\mathbf{b}_{-1} \cdot \mathbf{c} \cdots \mathbf{c} \cdot \mathbf{b}_{-n} \cdots \mathbf{b}_{-2})^{\mathbb{N}}$.

For every $k \in \{0, \dots, n\}$, let Q_{-k} be the unique intersection point of $\mathbb{T}'_{\bar{\mathbf{p}}^{-k}}$ with $W_{loc}^u(\underline{b}^{-k}; f')$, and let S'_{-k} be the preimage of S' by f'^k in $\mathbb{T}'_{\mathbf{b}_{-k} \cdots \mathbf{b}_{-1} \cdot \mathbf{c}^{\mathbb{N}}}$.

We notice that f' sends Q_{-k-1} to Q_{-k} for every $0 \leq k < n$.

Note that $Q_0 \in W_{loc}^u(\underline{b}; f') \cap \mathbb{T}'_{\bar{\mathbf{p}}^0}$ is close to $S' \in W_{loc}^u(\underline{b}; f') \cap \mathbb{T}'_{\mathbf{c}^{\mathbb{N}}}$ since $\bar{\mathbf{p}}^0 = (\mathbf{c}^n \cdot \mathbf{b}_{-n} \cdots \mathbf{b}_{-1})^{\mathbb{N}}$ is close to $\mathbf{c}^{\mathbb{N}}$. Similarly, $S'_{-k} \in W_{loc}^u(\underline{b}^{-k}; f') \cap \mathbb{T}'_{\mathbf{b}_{-k} \cdots \mathbf{b}_{-1} \cdot \mathbf{c}^{\mathbb{N}}}$ is close to $Q_{-k} \in W_{loc}^u(\underline{b}^{-k}; f') \cap \mathbb{T}'_{\bar{\mathbf{p}}^{-k}}$, for every $0 \leq k \leq n$.

Likewise, by C^1 -continuity of $(\mathbb{T}'_{\bar{\mathbf{a}}})_{\bar{\mathbf{a}}}$, $T_{Q_{-k}} \mathbb{T}'_{\bar{\mathbf{p}}^{-k}}$ is close to $T_{S'_{-k}} \mathbb{T}'_{\mathbf{b}_{-k} \cdots \mathbf{b}_{-1} \cdot \mathbf{c}^{\mathbb{N}}}$. Hence $Df'|T_{Q_{-k}} \mathbb{T}'_{\bar{\mathbf{p}}^{-k}}$ is close to $Df'|T_{S'_{-k}} \mathbb{T}'_{\mathbf{b}_{-k} \cdots \mathbf{b}_{-1} \cdot \mathbf{c}^{\mathbb{N}}}$. By Fact 3.2, this implies:

Fact 3.3. *When n is large, $(\sqrt{\|Df'^n|T_{Q_{-n}} \mathbb{T}'_{\bar{\mathbf{p}}^{-n}}\|})^{-1}$ is close to $\lambda_u = \|Df'^n|T_{S'} \mathbb{T}_{\mathbf{c}^{\mathbb{N}}}\|$.*

Let P_{-n} be the intersection point of $\mathbb{T}'_{\bar{\mathbf{p}}^{-n}}$ with the line $\{x = 0\}$. We are going to prove:

Lemma 3.4. *For a C^∞ -perturbation of f' small when n is large it holds:*

- *the point P_{-n} is $2n$ -periodic, and semi-parabolic (1 is an eigenvalue of $D_{P_{-n}} f'^{2n}$).*
- *The x -coordinate of P_{-n} is small.*

Before let us introduce a few notations and facts. For every $0 \leq k \leq 2n$, let $P_{k-n} := f'^k(P_{-n})$.

For $k \leq n$, we notice that P_{k-n} and Q_{k-n} are at most $(2/3)^{(1-2\epsilon)k}$ -distant. Consequently:

Fact 3.5. *When n is large, the point P_0 is close to S' and $\lambda'_u := (\sqrt{\|Df'^n|T_{P_{-n}} \mathbb{T}'_{\bar{\mathbf{p}}^{-n}}\|})^{-1}$ is close to $\lambda_u = \|Df'|T_{S'} \mathbb{T}_{\mathbf{c}^{\mathbb{N}}}\|$.*

Also we shall linearize f' at a neighborhood of S' by using:

Theorem 3.6 (Sternberg [Ste58]). *If a fixed point S of a C^∞ -diffeomorphism f' is non-resonant, then the restriction of f' to a neighborhood of S' is C^∞ -conjugate to its differential.*

We recall that a fixed point S is *non-resonant* if its eigenvalues $(\lambda_j)_j$ satisfy that for every $(n_j)_j \in \mathbb{N}$, for every i it holds:

$$|\lambda_i| \neq \prod_j |\lambda_j|^{n_j} \text{ if } \sum_j n_j \geq 2.$$

As $\lambda_{uu} > \lambda_u > 1$, in the present case, this condition is equivalent to say that λ_{uu} is not a power of λ_u . It comes:

Fact 3.7. *The non-resonance's condition is valid for an open and dense set of perturbations f' .*

Proof of Lemma 3.4. By Sternberg theorem, there exists a C^∞ -coordinate change Ψ of a neighborhood of S' , which fixes S' and so that the restriction A_0 of $\Psi \circ f' \circ \Psi^{-1}$ to $S' + (-1, 1)^2$ is:

$$A_0 : S' + (x, y) \mapsto S' + (\lambda_u \cdot x, \lambda_{uu} \cdot y).$$

For every $0 \leq k \leq n$, let P_{-n-k} be the preimage by $f'^k|_{\mathbb{T}'_{\mathbf{p}-n-k}}$ of P_{-n} . We notice that P_{-n-k} is close to S' when $k \leq n$ are large.

Let $m > 0$ be large and let m be large in function of n . Then P_{-m-n} is close to S' and so in the domain of Ψ . Put $(x_{-m-n}, y_{-m-n}) = \Psi(P_{-m-n}) - S'$.

Note that P_k does not belong to $N := \psi^{-1}(S' + (-x_{-n-m}, x_{-n-m}) \times (-y_{-n-m}, y_{-n-m}))$ for every $k \in \{-n-m, \dots, -1\}$. Put $N' := \psi^{-1}(S' + (-0.9 \cdot x_{-n-m}, 0.9 \cdot x_{-n-m}) \times (-0.9 \cdot y_{-n-m}, 0.9 \cdot y_{-n-m}))$

Hence it suffices to handle a perturbation f'' of f' supported by N , and given on N' by the following:

Lemma 3.8. *There exists an affine perturbation A of $\psi \circ f' \circ \psi^{-1}$, which is small when n is large and so that with $f'' := \psi^{-1} \circ A \circ \psi$ it holds:*

- (i) $f''^{n-m}(P_0) = P_{-n-m}$,
- (ii) $Df''^{n-m}(T_{P_0}\mathbb{T}'_{\mathbf{p}^0}) = T_{P_{-n-m}}\mathbb{T}'_{\mathbf{p}^n}$ and $\|Df'^m|_{T_{P_{-n-m}}\mathbb{T}'_{\mathbf{p}-n-m}}\| \cdot \|Df''^{n-m}|_{T_{P_0}\mathbb{T}'_{\mathbf{p}^0}}\| = \lambda_u'^n$,
- (iii) $f''^k(P_0)$ belongs to N' for every $0 \leq k \leq m-1$.

Indeed after such a perturbation, Lemma 3.4 is proved since P_n is $2n$ -periodic and semi-parabolic for f'' . Moreover the orbit of P_n shadows $(f'^k(P_n))_{0 \leq k \leq 2n}$ and so belongs to the continuation of $\mathbb{T}'_{\mathbf{p}^n}$. \square

Proof of Lemma 3.8. For every $k \in \{0, -n-m\}$, let $\tilde{P}_k := \psi(P_k)$ and $L_k := D_{P_k}\psi(T_{P_k}\mathbb{T}'_{\mathbf{p}^k})$.

Let $A_0 := (x, y) \mapsto (\lambda_u x, \lambda_{uu} y)$. Note that both L_0 and L_{-n-m} are far from being vertical.

Fact 3.9. *When n is large, there exists θ small so that with $R: (x, y) \mapsto (x, y + \theta x)$, the map $(R^{-1} \circ A_0 \circ R)^{n-m}$ sends L_0 to L_{-n-m} .*

Proof. We identify the space of non vertical lines of \mathbb{R}^2 to \mathbb{R} by associating to a line L its slope l . Let l_0 and l_{-n-m} be the slopes of respectively L_0 and L_{-n-m} . As both n and m are large, the slopes l_0 and l_{-n-m} are small. In this identification, the map $L \mapsto A_0(L)$ is $l \mapsto \frac{\lambda_{uu}}{\lambda_u} l$ and the map R is $l \mapsto l + \theta$. Hence $(R^{-1} \circ A_0 \circ R)^{n-m}(L_0) = L_{-n-m}$ is equivalent to the equation:

$$l_{-n-m} = \left(\frac{\lambda_{uu}}{\lambda_u}\right)^{n-m}(l_0 + \theta) - \theta \Leftrightarrow \theta = \frac{-l_0 + \left(\frac{\lambda_u}{\lambda_{uu}}\right)^{n-m} l_{-n-m}}{1 - \left(\frac{\lambda_u}{\lambda_{uu}}\right)^{n-m}}.$$

As n is large w.r.t. m , $\left(\frac{\lambda_u}{\lambda_{uu}}\right)^{n-m}$ is small. Also l_0 and l_{-n-m} are small. Hence there exists a solution θ to this equation, which is small when n is large w.r.t. m large. \square

We notice that since $n-m$ is large, and $n-m$ -first iterates of L_0 remain far from being vertical, the following is still close to λ_u :

$$\lambda_u'' := \sqrt[n-m]{\|D_{\tilde{P}_{-n-m}}\psi^{-1}|_{L_{n-m}}\| \cdot \|(R^{-1} \circ A_0 \circ R)^{n-m}|_{L_0}\| \cdot \|D\psi|_{T_{P_0}\mathbb{T}'_{\mathbf{p}^0}}\|}.$$

Also, since n is large w.r.t. m , the number $\lambda_u^{(3)} := \lambda_u'^{n/(n-m)} \cdot \|Df'^m|_{T_{P_{-n-m}}\mathbb{T}'_{\mathbf{p}-n-m}}\|^{-1/(n-m)}$ is close to λ_u' and so to λ_u .

Hence $A_1 = \frac{\lambda_u^{(3)}}{\lambda_u''} \cdot R^{-1} \circ A_0 \circ R$ is close to A_0 . Furthermore, $Df''^{n-m} := D_{\tilde{P}_{n-m}}\psi^{-1} \circ A_1^{n-m} \circ D_{P_0}\psi$ satisfies (ii).

To prove (i), it suffices to find S'' close to S' so that the affine map $A: (x, y) + S'' \mapsto A_1(x, y) + S''$ satisfies:

$$A^{n-m}(\tilde{P}_0) = \tilde{P}_{-n-m} \Leftrightarrow (A_1^{n-m} - id)(\tilde{P}_0 - S'') = \tilde{P}_{-n-m} - \tilde{P}_0$$

$$\Leftrightarrow S'' = -(A_1^{n-m} - id)^{-1}(\tilde{P}_{-n-m} - \tilde{P}_0) + \tilde{P}_0.$$

Note that S'' is indeed close to S' since $(A_1^{n-m} - id)^{-1}$ is small and \tilde{P}_0 is close to S' .

We notice that the $(n-m)$ -first iterates of \tilde{P}_0 belongs to an invariant segment by A^{-1} with endpoint S'' and \tilde{P}_{-n-m} , which is included in $[-x_{-n-m}, x_{-n-m}] \times [-y_{-n-m}, y_{-n-m}]$ by convexity. Hence (iii) is satisfied. \square

The following remark will be useful for the proof of Theorem 2.3.

Remark 3.10. In view of the proof, the affine map A' depends analytically on $\psi(P_0)$, $\psi(P_{-n-m})$, $D\psi(T_{P_0}\mathbb{T}'_{\bar{p}_0})$ and $D\psi(T_{P_{-n-m}}\mathbb{T}'_{\bar{p}_{-n-m}})$, λ_{uu} , λ_u , $\frac{\lambda'_u}{\lambda_u}$ and $\sqrt[m]{\|Df'^m|T_{P_{-n-m}}\mathbb{T}'_{\bar{p}_{-n-m}}\|}$.

Moreover when n is large w.r.t. m large, this dependence is C^∞ -small for all the parameters but for the dependence on $\psi(P_0) - S'$, $D\psi(T_{P_0}\mathbb{T}'_{\bar{p}_0})$ and $\frac{\lambda'_u}{\lambda_u}$ which remains C^∞ -bounded.

3.4 Conclusion for $\dim M = 2$

To achieve the proof of Theorem 2.2 in the 2-dimensional case it suffices to prove the following:

Proposition 3.11. *For f' satisfying the conclusion of Lemma 3.4, the map $f'^{2n}|_{\mathbb{T}'_{\bar{p}_{-n}}}$ is parabolic.*

To prove this proposition, we parametrize $\mathbb{T}'_{\bar{p}_k}$ by the first coordinate projection: $\mathbb{T}'_{\bar{p}_k} \rightarrow \mathbb{T}$.

Let $f_k: \mathbb{T} \rightarrow \mathbb{T}$ be the map $\mathbb{T}'_{\bar{p}_k} \rightarrow \mathbb{T}'_{\bar{p}_{k+1}}$ seen in the first coordinate projection. We recall that x_k denotes the first coordinate of P_k for every k .

In these coordinates, $f'^{2n}|_{\mathbb{T}'_{\bar{p}_{-n}}}$ is $F := f_{n-1} \circ \dots \circ f_1 \circ f_0 \circ \dots \circ f_{-n}$. Hence it suffices to show that F is parabolic. We are going to show:

Claim 3.12. *$F|[-3/2, 3/2]$ is C^2 -close to $h: x \mapsto \frac{x}{2x+1}$.*

Proof of Proposition 3.11. By Lemma 3.4, the point x_{-n} is a parabolic fixed point for F . As $F|[-3/2, 3/2]$ is C^2 -close to h , $D^2F(x_{-n})$ is non zero and x_{-n} is its unique fixed point in $[-3/2, 3/2]$. Moreover, since h sends $[-3/2, 3/2]$ onto $\mathbb{T} \setminus [3/8, 3/4]$, x_{-n} is the unique fixed point of F . \square

Proof of Claim 3.12. We recall that λ_u is the weak unstable eigenvalue of S' . In order to prove Claim 3.12, we propose the following renormalization trick based on the two following lemmas:

Lemma 3.13. *There exists h' C^2 -close to h so that $x \mapsto h(\lambda_u^n \cdot h'^{-1}(x))$ coincides with:*

$$f_{n-1}|[-2, 2] \circ \dots \circ f_1|[-2, 2] \circ f_0|[-2, 2]$$

Lemma 3.14. *The map $\Phi = \lambda_u^n \cdot h'^{-1} \circ f_{-1} \circ \dots \circ f_{-n+1} \circ f_{-n}|[-2, 2]$ is C^2 -close to the identity.*

Indeed by combining the two lemmas, the map $F|[-3/2, 3/2]$ is equal to the composition of h with a map Φ C^2 -close to the identity. \square

Proof of Lemma 3.13. When $f' = f$, we have $f_i(x) = h((3/2) \cdot h^{-1}(x))$ for every $0 \leq i \leq n-1$. Hence it suffices to extend this conjugacy for the C^2 -perturbation f' .

Fact 3.15. *Up to a conjugacy close to the identity when f' is close to f , we may assume that $W_{loc}^{uu}(S')$ is in $\{0\} \times (-1, 1)$. There is a continuous family of coordinates on $(\mathbb{T}'_{\bar{a}})_{\bar{a}}$, C^∞ -close to the first coordinate projection so that $\|Df'|T_0\mathbb{T}'_{\bar{c}\dots\bar{a}}\| = \lambda_u$ for every $\bar{a} \in \mathfrak{A}^{\mathbb{N}}$.*

Proof. The first assertion is clear. Let $\bar{a} \in \mathfrak{A}^{\mathbb{N}}$ and denote by A_n the point with zero x -coordinate in $\mathbb{T}_{\mathfrak{c}^n, \bar{a}}$. Note that A_n belongs to the strong unstable manifold of $W_{loc}^{uu}(S'; f')$ and satisfies $f'(A_n) = A_{n-1}$ for every n . As S' is projectively hyperbolic source, $(A_n)_n$ converges exponentially fast to S' and $(T_{A_n} \mathbb{T}'_{\mathfrak{c}^n \dots \bar{a}})_n$ converges exponentially fast to $T_{S'} \mathbb{T}'_{\mathfrak{c}^{\mathbb{N}}}$. Consequently $(\|Df'^n|_{T_0 \mathbb{T}'_{\mathfrak{c}^n \dots \bar{a}}}\|/\lambda_u^n)_n$ converges to a real number close to 1 when f' is C^2 -close to f .

Then for every $\bar{a} \in \mathfrak{A}^{\mathbb{N}}$ which does not start by \mathfrak{c} , the coordinate change $x \mapsto \frac{\|Df'^n|_{T_0 \mathbb{T}'_{\mathfrak{c}^n, \bar{a}}}\|}{\lambda_u^n} x$ on $\mathbb{T}'_{\mathfrak{c}^n, \bar{a}}$ satisfies the above fact. \square

Let H be the set of continuous families $(h_{\bar{a}})_{\bar{a} \in \mathfrak{A}^{\mathbb{N}}}$ of C^2 -diffeomorphisms $h_{\bar{a}}$ of $[-2, 2]$ into $\mathbb{T}'_{\bar{a}}$ so that $h_{\bar{a}}(0) = 0$ and $D_0 h_{\bar{a}} = 1$. We notice that H endowed with the following metric is complete:

$$d((h_{\bar{a}})_{\bar{a}}, (h'_{\bar{a}})_{\bar{a}}) \leq \max_{\bar{a} \in \mathfrak{A}^{\mathbb{N}}} \max_{[-2, 2]} \|D_x^2 h_{\bar{a}} - D_x^2 h'_{\bar{a}}\|.$$

We consider the operator:

$$\Psi: \bar{h} = (h_{\bar{a}})_{\bar{a} \in \mathfrak{A}^{\mathbb{N}}} \in H \mapsto (\Psi(\bar{h})_{\bar{a}})_{\bar{a}} \in H, \quad \text{with } \Psi(\bar{h})_{\bar{a}} = \begin{cases} h & \text{if } \bar{a} \in \mathfrak{B} \times \mathfrak{A}^{\mathbb{N}} \\ f' \circ h_{\bar{a}}(x/\lambda_u) & \text{if } \bar{a} \in \{\mathfrak{c}\} \times \mathfrak{A}^{\mathbb{N}} \end{cases}$$

where $\{\mathfrak{c}\} \times \mathfrak{A}^{\mathbb{N}}$ and $\mathfrak{B} \times \mathfrak{A}^{\mathbb{N}}$ are the subsets of $\mathfrak{A}^{\mathbb{N}}$ formed by sequences whose zero-coordinate is in respectively $\{\mathfrak{c}\}$ and \mathfrak{B} .

By the same argument as for Sternberg's linearization Theorem [Ste57], this operator has a contracting iterate for the distance d . Hence it has a unique fixed family $(h_{\bar{a}})_{\bar{a}}$. Note also that this operator depends continuously on f' . As for $f' = f$, the fixed family is the constant family $(h)_{\bar{a}}$, it comes that for f' C^2 -close to f , the fixed point $(h_{\bar{a}})_{\bar{a}}$ is C^2 -close the constant family $(h)_{\bar{a}}$.

Finally, we easily observe that $(h_{\bar{a}})_{\bar{a}}$ displays the wished conjugacy. \square

Proof of Lemma 3.14. As the x -coordinate x_{-n} of P_{-n} is close to 0, the point $h'^{-1}(x_{-n})$ is small and the derivative $Dh'^{-1}(x_{-n})$ is close to 1. As x_{-n} is a parabolic fixed point of F , the map $\Phi = h'^{-1} \circ F$ sends x_{-n} to a point close to 0 and its derivative at x_{-n} is close to 1. Hence it suffices to show that the second derivative of Ψ is small:

$$D^2 \Phi = \lambda_u^n \cdot D^2 h'^{-1} \cdot (Df_{-1} \cdots Df_{-n})^2 + \sum_{i=1}^n \lambda_u^n \cdot Dh'^{-1} \cdot Df_{-1} \cdots Df_{-i+1} \cdot D^2 f_{-i} (Df_{-i-1} \cdots Df_{-n})^2$$

The first term $\lambda_u^n \cdot D^2 h'^{-1} \cdot (Df_{-1} \cdots Df_{-n})^2$ is of the order of λ_u^{-m} which is small.

Let us bound the sum. The derivatives $D^2 f_{-i}$ are all small, whereas $|\lambda_u^n \cdot Df_{-1} \cdots Df_{-i+1} \cdot (Df_{-i-1} \cdots Df_{-n})^2|$ is of the order of $|Df_{-i-1} \cdots Df_{-n}|$ which is exponentially small w.r.t $n-i$, with factor close to $2/3$. Hence the sum is small. \square

3.5 Conclusion for the general case $\dim M \geq 3$

The proof follows the same scheme. We only relate the substantial modifications to make.

The normally expanded fibration by circles is replaced by a normally hyperbolic one.

Instead of considering the map $f : (x, y) \in \mathbb{A} \mapsto (f_y(x), \sigma(y)) \in \mathbb{A}$ as defined in section 3.2, we consider the map

$$F_b : (x, y, z, (w_i)_i) \in \mathbb{A} \times [-2, 2]^{d-2} \mapsto (f_y(x), \sigma(y) - b \cdot z, b \cdot y, b \cdot (w_i)_i)$$

with $b \in (0, (2/3)^2)$ small. We notice that F_b is a diffeomorphism onto its image for $b > 0$ small since it is the skew product of $(x, (w_i)_i) \mapsto (f_y(x), b \cdot (w_i)_i)$ over the a generalized Hénon map $(y, z) \mapsto (\sigma(y) - b \cdot z, b \cdot y)$, which are both invertible. Here is a counterpart of Fact 3.1

Fact 3.16. *We can embed the dynamics F_b into any manifold of dimension M of dimension d .*

Proof. We notice that it suffices to build an extension of F_b which is a diffeomorphism of $\mathbb{T} \times [-3, 3]^{d-1}$ which coincides with the identity at a neighborhood of the boundary of $\partial(\mathbb{T} \times [-3, 3]^{d-1}) = \mathbb{T} \times \partial([-3, 3]^{d-1})$.

It is easy to extend by the identity a generalized Hénon map away of a large ball. Similarly, there exists a C^∞ -diffeomorphism H of $[-3, 3]^{d-1}$ so that:

- the restriction of H to $[-2, 2]^{d-1}$ is $(y, z) \mapsto (\sigma(y) - b \cdot z, b \cdot y, b \cdot (w_i)_i)$,
- the restriction of H to $[-3, 3]^{d-1} \setminus [-5/2, 5/2]^{d-1}$ is the identity.

Let $(f_{y,z,w})_{(y,z,w) \in [-3,3]^{d-1}}$ be a C^∞ -family of circle diffeomorphism $x \mapsto f_{y,z,w}(x)$ so that:

- for $(y, z, w) \in [-1, 1]^{d-1}$ it holds $f_{y,z,w} = f_y$,
- $f_{y,z,w} = id_{\mathbb{T}}$ if $(y, z, w) \in [-3, 3]^{d-1} \setminus [-5/2, 5/2]^{d-1}$.

We notice that $(x, y, z, w) \mapsto (f_{y,z,w}(x), H(y, w, z))$ satisfies the requested property. \square

We notice that for $b = 0$ the map F_0 is equal to the product $(f, 0)$ of f with 0. Furthermore, the lamination $\mathcal{L} = (\mathbb{T}_{\bar{\alpha}})_{\bar{\alpha} \in \mathfrak{A}^{\mathbb{N}}}$ is 2-normally hyperbolic for f , with as central direction the first coordinate, as strongly expanded direction the second coordinate, and as strongly contracted direction the $n - 3$ last coordinates.

To study the persistence of \mathcal{L} , as the map F_0 is not invertible, we shall consider the inverse limit $\overleftarrow{\mathcal{L}} = (\mathbb{T}_{\bar{\alpha}})_{\bar{\alpha} \in \mathfrak{A}^{\mathbb{Z}}}$ of \mathcal{L} , with $\mathbb{T}_{\bar{\alpha}}$ equal to $\mathbb{T}_{\bar{\alpha}}$ with $\bar{\alpha} \in \mathfrak{A}^{\mathbb{N}}$ the sequence of non-negative coordinates of $\bar{\alpha}$. As F_0 sends diffeomorphically $\mathbb{T}_{\bar{\alpha}}$ onto $\mathbb{T}_{\bar{\sigma}(\bar{\alpha})}$ with $\bar{\sigma}$ the shift of $\mathfrak{A}^{\mathbb{Z}}$ it holds:

Theorem 3.17 ([Ber10]). *For every $b > 0$ small, for every F' C^2 -close to F_b , there exists a continuous family $(\mathbb{T}'_{\bar{\alpha}})_{\bar{\alpha} \in \mathfrak{A}^{\mathbb{Z}}}$ of disjoint C^2 -circles such that:*

$$F'(\mathbb{T}'_{\bar{\alpha}}) = \mathbb{T}'_{\bar{\sigma}(\bar{\alpha})} \quad , \quad \forall \bar{\alpha} \in \mathfrak{A}^{\mathbb{Z}}.$$

and $\mathbb{T}'_{\bar{\alpha}}$ is C^2 -close to $\mathbb{T}'_{\bar{\alpha}}$ with $\bar{\alpha}$ the canonical projection of $\bar{\alpha}$ into $\mathfrak{A}^{\mathbb{N}}$.

Let us denote by S' the hyperbolic continuation of S . Let λ_u be the weak unstable eigenvalue of S' .

We notice that a local stable manifold $W_{loc}^s(S'; F')$ of S' is C^r -close to $S' + \{0\} \times \{0\} \times (-1, 1)^{d-2}$. For every $\underline{b} \in \mathfrak{B}^{\mathbb{Z}^-}$ we denote by $W_{loc}^u(\underline{b}; F')$ the local unstable manifold equal to the hyperbolic continuation of the local unstable manifold $W_{loc}^u(\underline{b}; f) \times \{0\}^{d-2}$ of F_0 . We denote by $\underline{b} \cdot \mathbf{c}^{\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$ the sequence with \underline{b} as negatives entries and with $\mathbf{c}^{\mathbb{N}}$ as non negative ones.

By Fact 1.14, for $\epsilon > 0$ small enough, and then F' C^1 -close to F_b , it holds:

Fact 3.18. *There exists $\underline{b} = (b_i)_{i < 0} \in \mathfrak{B}^{\mathbb{Z}^-}$ so that $W_{loc}^u(\underline{b}; F')$ intersects $W_{loc}^s(S'; F')$ at a point $Q \in \mathbb{T}'_{\underline{b} \cdot \mathbf{c}^{\mathbb{N}}}$, and with $Q_{-n} := F'^{-n}(Q)$ it holds:*

$$\frac{1}{n} \log \|DF^n|_{T_{Q_{-n}} \mathbb{T}'_{\bar{\sigma}^{-n}(\underline{b} \cdot \mathbf{c}^{\mathbb{N}})}}\| \rightarrow -\log \lambda'_u$$

For every n large, we define the $2n$ -periodic point $\overleftarrow{\mathbf{p}}^0 = (\mathbf{c} \cdots \mathbf{c} \cdot \mathbf{b}_{-n} \cdots \mathbf{b}_{-1})^{\mathbb{Z}}$ with n first entries equal to \mathbf{c} and n first negative entries equal to $\mathbf{b}_{-n} \cdots \mathbf{b}_{-1}$ given by the above fact.

We put also for every $k \in \mathbb{Z}$, $\overleftarrow{\mathbf{p}}^k := \sigma^k(\overleftarrow{\mathbf{p}}^0)$. Let P_{-n} be the point of $\mathbb{T}'_{\overleftarrow{\mathbf{p}}^{-n}}$ with x -coordinate equal to 0. Put $P_{-n+k} := F'^k(P_{-n})$ for every $k \in \mathbb{Z}$. Here is the counterpart of Claim 3.4:

Claim 3.19. *For a C^∞ -perturbation of F' small when n is large it holds:*

- *the point P_{-n} is $2n$ -periodic, and semi-parabolic (it has one eigenvalue $= 1$).*
- *The x -coordinate of P_{-n} is small.*

Proof. To prove this claim, let us first note that the same proof as for Fact 3.5 shows:

Fact 3.20. *When n is large, the point P_0 is close to Q and $\lambda'_u := (\sqrt[n]{\|DF'^n|T_{P_{-n}}\mathbb{T}'_{\overleftarrow{\mathbf{p}}^{-n}}\|})^{-1}$ is close to $\lambda_u = \|DF'|T_{S'}\mathbb{T}_{\mathbb{C}^N}\|$.*

Furthermore, up to a small perturbation, we can assume the eigenvalue of S' non-resonant. Then by Sternberg theorem [Ste58], F' is linearizable on a neighborhood of S' . We notice that by pulling back the conjugacy, we can assume it well defined on a neighborhood of the local stable manifold $W_{loc}^s(S', F')$. In other words, there exists a C^∞ -smooth map ψ from a small neighborhood D of $W_{loc}^s(S', F')$ onto an open subset of $(-1, 1)^d + S'$ so that $A_0 := \psi \circ F' \circ \psi^{-1}$ has the form:

$$A_0: S' + (x, y, z, w) \mapsto S' + (\lambda_u \cdot x, \lambda_{uu} \cdot y, \beta(z, w)),$$

with β a linear function. Let $m \geq 0$ be large but small w.r.t. n . Then P_{-n-m} is close to S' and so belongs to D . Furthermore, $T_{P_{-n-m}}\mathbb{T}'_{\overleftarrow{\mathbf{p}}^{-n-m}}$ is close to $T_{S'}\mathbb{T}'_{\mathbb{C}^Z}$.

We denote by (x_{-n-m}, y_{-n-m}) the two first coordinates of P_{-n-m} .

Note that P_k does not belong to $N := \psi^{-1}(S' + (-x_{-n-m}, x_{-n-m}) \times (-y_{-n-m}, y_{-n-m}) \times (-1, 1)^{d-2})$ for every $k \in \{-n-m, \dots, -1\}$. Put $N' := \psi^{-1}(S' + (-0.9 \cdot x_{-n-m}, 0.9 \cdot x_{-n-m}) \times (-0.9 \cdot y_{-n-m}, 0.9 \cdot y_{-n-m}) \times (-1, 1)^{d-2})$.

Then it suffices to handle a perturbation supported by N , and given on N' by the same statement of Lemma 3.8 with F' instead of f' . Actually the proof is the same, but the expansion is now replaced by hyperbolicity which enables to find the neat perturbation as a fixed point. \square

To accomplish the proof of Theorem A in any dimension we show likewise:

Proposition 3.21. *Under the conclusion of Claim 3.19, the map $f'^{2n}|_{\mathbb{T}''_{\overleftarrow{\mathbf{p}}^{-n}}}$ is parabolic.*

Proof. The proof follows the same argument as the one of Proposition 3.11.

First we endow $\mathbb{T}'_{\overleftarrow{\mathbf{p}}^k}$ with the coordinate given by the first coordinate projection:

$$\mathbb{T}'_{\overleftarrow{\mathbf{p}}^k} \rightarrow \mathbb{T}.$$

Let $f_k: \mathbb{T} \rightarrow \mathbb{T}$ be the map $F': \mathbb{T}'_{\overleftarrow{\mathbf{p}}^k} \rightarrow \mathbb{T}''_{\overleftarrow{\mathbf{p}}^{k+1}}$ seen in the first coordinate projection.

In these coordinates, $F'^{2n}|_{\mathbb{T}'_{\overleftarrow{\mathbf{p}}^{-n}}}$ is $F := f_{n-1} \circ \dots \circ f_1 \circ f_0 \circ \dots \circ f_{-n}$. Hence it suffices to show that F is parabolic.

In order to show to do so, we notice that it is a consequence of Claim 3.12 by words for words the same argument in this new setting. On the other hand, Claim 3.12 is a consequence of Lemmas 3.13 and 3.14. The proof of Lemma 3.14 is exactly the same for this setting. The proof of Lemma 3.13 in this setting is done likewise by considering the following operator:

$$\Psi: \overleftarrow{h} = (h_{\overleftarrow{\mathbf{a}}})_{\overleftarrow{\mathbf{a}} \in \mathfrak{A}^{\mathbb{Z}}} \mapsto (\Psi(\overleftarrow{h})_{\overleftarrow{\mathbf{a}}})_{\overleftarrow{\mathbf{a}}}, \text{ with } \Psi(\overleftarrow{h})_{\overleftarrow{\mathbf{a}}} = \begin{cases} h & \text{if } \overleftarrow{\mathbf{a}} \in \mathfrak{A}^{\mathbb{Z}^-} \times \mathfrak{B} \times \mathfrak{A}^{\mathbb{Z}^+} \\ f' \circ \phi_{\overleftarrow{\mathbf{a}}}(x/\lambda_u) & \text{if } \overleftarrow{\mathbf{a}} \notin \mathfrak{A}^{\mathbb{Z}^-} \times \{\mathbf{c}\} \times \mathfrak{A}^{\mathbb{Z}^+} \end{cases}$$

where $\mathfrak{A}^{\mathbb{Z}^-} \times \mathfrak{B} \times \mathfrak{A}^{\mathbb{Z}^+}$ and $\mathfrak{A}^{\mathbb{Z}^-} \times \{\mathfrak{c}\} \times \mathfrak{A}^{\mathbb{Z}^+}$ are the subsets of $\mathfrak{A}^{\mathbb{Z}}$ formed by sequences whose 0-coordinate is in respectively \mathfrak{B} and $\{\mathfrak{c}\}$. \square

4 Local density of families which display robustly a parabolic circle map

This section is devoted to the proof of Theorem 2.3. Let $k \geq 1$ and $\infty > r \geq 2$. We wish to construct an open set \hat{U} of families $\hat{f} = (f_a)_{a \in \mathbb{R}^k}$ and $\alpha > 0$ so that there exists a C^r -dense set $\hat{D} \subset \hat{U}$ formed by C^∞ -families $(f'_a)_a$ preserving a normally expanded circle bundle $((\mathbb{T}'_{\bar{\mathfrak{a}}}(a))_{\bar{\mathfrak{a}} \in \mathfrak{A}^{\mathbb{N}}})_a$ so that the following property holds:

There exists an open covering $(U_i)_i$ of $[-\alpha, \alpha]^k$ and an integer family $(p_i)_i$, so that for every i , there exist a p_i -periodic sequence $\bar{\mathfrak{a}}_i \in \mathfrak{A}^{\mathbb{N}}$ for which $f'_a|_{\mathbb{T}'_{\bar{\mathfrak{a}}_i}(a)}$ is parabolic whenever $a \in U_i$.

For this end, we are going to push forward the proof of Theorem 2.2, by replacing the λ -blender by a C^r - λ -parablender.

4.1 Definition of the family \hat{f}

Let $\mathfrak{A} := \mathfrak{B} \cup \mathfrak{C}$, where $\mathfrak{C} := \{-1, 1\}^k$ and where \mathfrak{B} equals to $\Delta_r \times \Delta_{r-1}$ with Δ_r defined in Example 1.20. Let $(I_a)_{a \in \mathfrak{A}}$ be disjoint non-trivial segments in $(-1, 1)$ and σ a C^∞ -map which sends affinely each I_a onto $[-1, 1]$, with derivatives $> (3/2)^{r+2}$. Put $D := \cup_{a \in \mathfrak{A}} I_a$ and consider:

$$f_a : (x, y) \in D \times [-3, 3] \mapsto (f_a|_a(x), \sigma(y)) \quad \text{if } y \in I_a$$

with:

$$(H_1) \quad f_{\mathfrak{c}}|_a(x) = \frac{3}{2} \cdot \frac{x}{x+1} \quad \text{if } \mathfrak{c} \in \mathfrak{C}.$$

(H₂) For every $\mathfrak{c} \in \mathfrak{C}$ if $y_{\mathfrak{c}} \in I_{\mathfrak{c}}$ is the fixed point of σ , then the derivative $D_{x_{\mathfrak{c}}} \sigma$ is not a power of $3/2$.

$$(H_3) \quad f_{\mathfrak{a}}|_a(x) = \left(\frac{2}{3} \cdot \exp(\epsilon \cdot P_{\delta'}(a)) \cdot x + \frac{P_{\delta}(a)}{3}\right) \quad \text{if } \mathfrak{a} = (\delta, \delta') \in \Delta_r \times \Delta_{r-1}.$$

The maximal invariant set $\cap_{j \geq 0} \sigma^{-j}(\cup_{a \in \mathfrak{A}} I_a)$ is expanding and identified to $\mathfrak{A}^{\mathbb{N}}$. In this identification σ is the shift map $\bar{\mathfrak{a}} = (\mathfrak{a}_i)_{i \geq 0} \mapsto (\mathfrak{a}_{i+1})_{i \geq 0}$.

The fibration $(\mathbb{T}_{\bar{\mathfrak{a}}})_{\bar{\mathfrak{a}} \in \mathfrak{A}^{\mathbb{N}}}$ is left invariant by f_a with $\mathbb{T}_{\bar{\mathfrak{a}}} = \mathbb{T}_{\bar{\mathfrak{a}}} \times \{\bar{\mathfrak{a}}\}$. Moreover for a small, it is $r+1$ -normally expanded. Actually, for α small, the same occurs for the fibration $(\cup_{a \in (-\alpha, \alpha)^k} \{a\} \times \mathbb{T}_{\bar{\mathfrak{a}}})_{\bar{\mathfrak{a}}}$ and the dynamics $(a, z) \mapsto (a, f_a(z))$.

Hence by Theorem 1.30, for every $\alpha > 0$ there exists a C^r -neighborhood \hat{U} of $(f_a)_a$, so that for every $\hat{f}' = (f'_a)_a \in \hat{U}$, for every $a \in [-\alpha, \alpha]^k$ there exists a fibration $(\mathbb{T}'_{\bar{\mathfrak{a}}}(a))_{\bar{\mathfrak{a}} \in \mathfrak{A}^{\mathbb{N}}}$ left invariant by f'_a . Moreover its fibers are of class C^{r+2} and depend C^{r+2} on the parameter a . In other words, the family $(\cup_{a \in (-\alpha, \alpha)^k} \{a\} \times \mathbb{T}'_{\bar{\mathfrak{a}}}(a))_{\bar{\mathfrak{a}}}$ is a continuous family of C^{r+2} -submanifolds.

4.2 Simple dynamical property of \hat{f}

South-North dynamics on $\mathbb{T}_{\mathfrak{C}^{\mathbb{N}}}$ For every $\mathfrak{c} \in \mathfrak{C}$, we notice that the point $S_{\mathfrak{c}} := (0, y_{\mathfrak{c}})$ is a source fixed by f_a and it is not resonant for every a by $(H_1 - H_2)$.

Hence for every $\alpha > 0$, for \hat{U} sufficiently C^r -small, for every $(f'_a)_a \in \hat{U}$, for every $a \in [-\alpha, \alpha]^k$, the source $S_{\mathfrak{c}}$ persists as a source $S'_{\mathfrak{c}}|_a$ for f'_a and is not resonant: if $1 < \lambda_{uc}(a) < \lambda_{uuc}(a)$ are

its eigenvalues, it holds:

$$\lambda_{uc}(a) \neq \lambda_{uc}^j(a), \quad \forall j \geq 0, \quad \forall a \in (-\alpha, \alpha)^k$$

When $(f'_a)_a$ is of class C^∞ , this implies :

Theorem 4.1 (Sell, Takens [Sel85, Tak71]). *For every $(f'_a)_a \in \hat{U}$ of class C^∞ , for every $\mathfrak{c} \in \mathfrak{C}$, there exists a C^{r+2} -family of charts $(\psi_{\mathfrak{c}a})_{a \in [-\alpha, \alpha]^k}$ from a neighborhood of $S'_{\mathfrak{c}a}$ and so that for every $a \in [-\alpha, \alpha]^k$ it holds:*

$$\psi_{\mathfrak{c}a} \circ f_a \circ \psi_{\mathfrak{c}a}^{-1} = S'_{\mathfrak{c}a} + (x, y) \mapsto S'_{\mathfrak{c}a} + (\lambda_{uc}(a) \cdot x, \lambda_{uc}(a) \cdot y).$$

North-South dynamics on $(\mathbb{T}_{\mathfrak{b}})_{\mathfrak{b} \in \mathfrak{B}^\mathbb{N}}$ We notice that the maximal invariant of $\cup_{\mathfrak{b} \in \mathfrak{B}} [-3, 3] \times I_{\mathfrak{b}}$ is the λ - C^r -parablender B given by Example 1.25. For every $\hat{f}' = (f'_a)_a$ C^r -close to \hat{f} , for every $\underline{\mathfrak{b}} = (\mathfrak{b}_i)_{i \leq 0} \in \mathfrak{B}^{\mathbb{Z}^-}$, we define:

$$W_{loc}^u(\underline{\mathfrak{b}}; f'_a) = \cap_{n \geq 1} f_a'^n([-3, 3] \times I_{\mathfrak{b}_{-n}}).$$

A direct consequence of the λ - C^r -parablender property of B is the following:

Fact 4.2. *For a smaller C^r -neighborhood \hat{U} of \hat{f} , there exists $\alpha > 0$, so that for every n large enough, for every $\hat{f}' \in \hat{U}$, for every $a_0 \in [-\alpha, \alpha]^k$, for every $\mathfrak{c} \in \mathfrak{C}$, there exist $\underline{\mathfrak{b}} = (\mathfrak{b}_i)_{i < 0} \in \mathfrak{B}^{\mathbb{Z}^-}$ and a C^r -curve $(S'_{0a})_a \in (W_{loc}^u(\underline{\mathfrak{b}}, f'_a) \cap \mathbb{T}'_{\mathfrak{c}^\mathbb{N}}(a))_a$ satisfying:*

$$J_{a_0}^r(S'_{0a})_a = J_{a_0}^r(S'_{\mathfrak{c}a})_a \quad \text{and} \quad \lim_{n \rightarrow \infty} J_{a_0}^{r-1}(\|Df_a'^n|_{T_{S'_{-na}} \mathbb{T}_{\mathfrak{b}_{-n} \cdots \mathfrak{b}_{-1} \mathfrak{c}^\mathbb{N}}(a)}\|^{-\frac{1}{n}})_a = J_{a_0}^{r-1}(\lambda_{uc}(a))_a,$$

with S'_{-na} the preimage by $f_a'^n|_{\mathbb{T}'_{\mathfrak{b}_{-n} \cdots \mathfrak{b}_{-1} \mathfrak{c}^\mathbb{N}}(a)}$ of S'_{0a} and $\mathfrak{b}_{-n} \cdots \mathfrak{b}_{-1} \mathfrak{c}^\mathbb{N}$ the sequence with n -first terms $\mathfrak{b}_{-n} \cdots \mathfrak{b}_{-1}$ and then equal to \mathfrak{c} .

4.3 Creation of semi-parabolic point with C^r -degenerate unfolding

Given n large, $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$, we denote by $\bar{\mathfrak{p}}^0(n, a_0, \mathfrak{c}, \hat{f}')$ the $2n$ -periodic point $(\mathfrak{c} \cdots \mathfrak{c} \cdot \mathfrak{b}_{-n} \cdots \mathfrak{b}_{-1})^\mathbb{N}$ of $\mathfrak{A}^\mathbb{N}$, where $\mathfrak{b}_{-n} \cdots \mathfrak{b}_{-1}$ are the n last symbols of $\underline{\mathfrak{b}}$ given by Fact 4.2.

Lemma 4.3. *For every $\eta > 0$ and for every $\hat{f}' \in \hat{U}$ of class C^∞ , there exist $n \geq 1$ and $\eta' > 0$ so that for every $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$ there exists a C^r - η -small perturbation \hat{f}'' of \hat{f}' supported in $([-\eta', \eta']^k + a_0) \times \mathbb{T} \times I_{\mathfrak{c}}$, of class C^{r+1} and satisfying the following property.*

With $\bar{\mathfrak{p}}^0 = \bar{\mathfrak{p}}^0(n, a_0, \mathfrak{c}, \hat{f}')$ the above $2n$ -periodic point of $\mathfrak{A}^\mathbb{N}$, for every $a \in [-\frac{2}{3}\eta', \frac{2}{3}\eta']^k + a_0$, the map f_a'' displays a semi-parabolic $2n$ -periodic point in $\mathbb{T}_{\bar{\mathfrak{p}}^0}''(a)$.

Proof. The proof follows the same scheme as for Lemma 3.4.

First let us recall that for every $k \in \mathbb{Z}$, we denote by $\bar{\mathfrak{p}}^k$ the $2n$ periodic point in $\sigma^k(\{\bar{\mathfrak{p}}^0\})$. For every $k \in \{0, \dots, n\}$, let Q_{-ka} be the unique intersection point of $\mathbb{T}'_{\bar{\mathfrak{p}}^{-k}}(a)$ with $W_{loc}^u(\underline{\mathfrak{b}}^{-k}; f'_a)$ with $\underline{\mathfrak{b}}^{-k} = (\cdots \mathfrak{b}_{-k-2} \cdot \mathfrak{b}_{-k-1}) \in \mathfrak{B}^{\mathbb{Z}^-}$. By Fact 4.2 and the same argument of the proof of Lemma 3.4, we deduce:

Fact 4.4. *When n is large, $J_{a_0}^r(Q_{0a})_a$ is close to $J_{a_0}^r(S'_{\mathfrak{c}a})_a$ and $J_{a_0}^{r-1}(\|Df_a'^n|_{T_{Q_{-na}} \mathbb{T}_{\bar{\mathfrak{p}}^{-n}}}\|^{-\frac{1}{n}})_a$ is close to $J_{a_0}^{r-1}(\lambda_{uc}(a))_a$. Moreover the convergences are uniform among $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$.*

By hyperbolic continuation, the source $S'_{\mathfrak{c}a}$ and its weak unstable eigenvalue $\lambda_{u\mathfrak{c}}(a)$ depend C^∞ on a . Also, the intersection point $Q_{-k a}$ of $\mathbb{T}'_{\mathbb{P}^{-k}}(a)$ with $W_{loc}^u(\underline{b}^{-k}; f'_a)$ depends C^{r+2} on a with C^{r+2} -norm bounded independently on k since it is the case for $\mathbb{T}'_{\mathbb{P}^{-k}}(a)$ and $W_{loc}^u(\underline{b}^{-k}; f'_a)$. Consequently, the tangent space of $\mathbb{T}'_{\mathbb{P}^{-k}}(a)$ at $Q_{-k a}$ depends C^{r+1} on a with C^{r+1} -norm bounded independently on k . Hence it holds:

Fact 4.5. *When n is large, there exists $M = M(\hat{f}') > 0$ independent of n , $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$ so that the C^{r+2} -norm of $a \in [-\alpha, \alpha]^k \mapsto (Q_{0a}, S'_{\mathfrak{c}a})$ is bounded by M and the C^{r+1} -norm of $a \in [-\alpha, \alpha]^k \mapsto \|Df'_a|T_{Q_{-n}a}\mathbb{T}'_{\mathbb{P}^{-n}}\|^{-\frac{1}{n}} - \lambda_{u\mathfrak{c}}(a)$ is bounded by M .*

Let $P_{-n}a$ be the point in $\mathbb{T}'_{\mathbb{P}^{-n}}(a)$ with x -coordinate equal to 0. For every $k \in \mathbb{Z}$, let $P_{-n+k}a$ be its image f_a^k . As $Df'_a|T'_{\mathbb{P}^{-k}}(a) \cap \{x \in [-3, 3]\}$ is contracting with factor $\frac{2}{3}\exp(2\epsilon(1 + O(\alpha)))$ for every $k \in [-n, -1]$, the map $J^{r+2}(f_a|T'_{\mathbb{P}^{-k}}(a) \cap \{x \in [-3, 3]\})$ is as well contracting with factor $\frac{2}{3}\exp(2\epsilon(1 + O(\alpha)))$.

With $\lambda'_u(a) := \|Df'_a|T_{P_{-n}a}\mathbb{T}'_{\mathbb{P}^{-n}}\|^{-\frac{1}{n}}$, this implies:

Fact 4.6. *For every $\hat{f}' \in \hat{U}$ of class C^∞ , for every $a_0 \in (-\alpha, \alpha)^k$, $\mathfrak{c} \in \mathfrak{C}$, when n is large, $J_{a_0}^r(P_{0a})_a$ is close to $J_{a_0}^r(S_{\mathfrak{c}a})_a$ and $J_{a_0}^{r-1}(\lambda'_u(a))_a$ is close to $J_{a_0}^{r-1}(\lambda_{u\mathfrak{c}}(a))_a$.*

Moreover the convergences are uniform among $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$.

Furthermore, the C^{r+2} -norm of $a \in [-1, 1]^k \mapsto S_{\mathfrak{c}a} - P_{0a}$ and the C^{r+1} -norm of $a \in [-1, 1]^k \mapsto (\lambda'_u(a), \lambda_{u\mathfrak{c}}(a))$ are bounded by $M = M(\hat{f}')$ independent of a_0 , n and \mathfrak{c} .

Since $(P_{ka})_{0 \leq k \leq n-1}$ is not in $\{y \in I_{\mathfrak{c}}\}$, but P_{na} and $S_{\mathfrak{c}a}$ are in the interior of $\{y \in I_{\mathfrak{c}}\}$, by the latter fact, it holds:

Fact 4.7. *There exists $\eta' > 0$ arbitrarily small and independent of $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$, so that after a C^r - $\eta/2$ -small perturbation localized in $([-\eta', \eta']^k + a_0) \times \{y \in I_{\mathfrak{c}}\}$ of \hat{f}' , we can assume that $\partial_a^r \lambda'_u(a)$ and $\partial_a^r \lambda_{u\mathfrak{c}}(a)$ are zero for every $a \in a_0 + [-\frac{3}{4}\eta', \frac{3}{4}\eta']^k$.*

We recall that by Theorem 4.1, there exists a C^{r+2} -family $(\Psi_a)_a$ of maps Ψ_a , each of which sends a neighborhood of $S'_{\mathfrak{c}a}$ onto $S'_{\mathfrak{c}a} + (-\epsilon, \epsilon)^2$ and satisfies:

$$\Psi_a \circ f'_a \circ \Psi_a^{-1} : S_{\mathfrak{c}a} + (x, y) \mapsto S_{\mathfrak{c}a} + (\lambda_u(a) \cdot x, \lambda_{uu}(a) \cdot y).$$

Hence by a C^{r+2} -small perturbation we can assume that $(\Psi_a)_a$ is of class C^∞ .

Let $m > 0$ be large and let n be large in function of n . Then $P_{-m-n}a$ is close to $S'_{\mathfrak{c}a}$ and so $P_{-m-n}a$ is in the domain of Ψ_a for $m \leq n$ large enough for every $a \in [-\alpha, \alpha]^k$. Furthermore, $J_{a'_0}^{r+1}(P_{-m-n}a)_a$ is close to $J_{a'_0}^{r+1}(S'_{\mathfrak{c}a})_a$ and $J_{a'_0}^{r+1}(T_{P_{-m-n}a}\mathbb{T}'_{\mathbb{P}^{-n-m}}(a))_a$ is close to $J_{a'_0}^{r+1}(T_{S'_{\mathfrak{c}a}}\mathbb{T}'_{\mathbb{P}^{-n}}(a))_a$ for every $a'_0 \in [-\alpha, \alpha]^k$. Note that m is bounded from below independently of a_0 , $\mathfrak{c} \in \mathfrak{C}$ and n large.

On the other hand, when n is large, $J_{a_0}^r(P_{0a})_a$ is close to $J_{a_0}^r(S_{\mathfrak{c}a})_a$ and $J_{a_0}^r(T_{P_{0a}}\mathbb{T}_{\mathbb{P}^0}(a))_a$ is close to $J_{a_0}^r(T_{S_{\mathfrak{c}a}}\mathbb{T}_{\mathbb{P}^0}(a))_a$; both of them uniformly w.r.t. a_0 and \mathfrak{c} . By continuity, when $m < n$ are fixed, for $\eta' > 0$ small enough independent of \mathfrak{c} and $a_0 \in [-\alpha, \alpha]^k$, so that for every $a'_0 \in a_0 + [-\eta', \eta']^k$, the jet $J_{a'_0}^r(P_{0a})_a$ remains close to $J_{a'_0}^r(S_{\mathfrak{c}a})_a$ and $J_{a'_0}^r(T_{P_{0a}}\mathbb{T}_{\mathbb{P}^0}(a))_a$ remains close to $J_{a'_0}^r(T_{S_{\mathfrak{c}a}}\mathbb{T}_{\mathbb{P}^0}(a))_a$.

By proceeding following the algorithm provided by Lemma 3.8, for every $a \in a_0 + [-\eta, \eta]^k$, we handle a C^{r+1} -perturbation f''_a of f'_a which is supported by a neighborhood of $S'_{\mathfrak{c}a_0}$ and disjoint to $(P_{ia})_{-n-m \leq i \leq 0}$ so that $f_a^{m-m}(P_{0a})$ is sent to $P_{-n-m}a$ and P_{0a} is a semi-parabolic point of period $2n$.

Moreover, by Remark 3.10, this perturbation depends analytically on $\Psi_a(P_{0a}), \Psi_a(P_{-n-ma}), D\Psi_a(T_{P_{0a}}\mathbb{T}'_{\mathbb{P}^0})$ and $D\Psi_a(T_{P_{-n-ma}}\mathbb{T}'_{\mathbb{P}^{-n-m}}), \lambda_{uu}(a), \lambda_u(a), \frac{\lambda'_u(a)}{\lambda_u(a)}$ and $\sqrt[m]{\|Df'^m|T_{P_{-n-ma}}\mathbb{T}'_{\mathbb{P}^{-n-m}}\|}$.

As $(\Psi_a)_a$ and $(f'_a)_a$ are of class C^∞ , the family $(f''_a)_a$ is of class C^∞ .

As the dependence of these variables is C^{r+1} -bounded on a when m is fixed and n large, the family $(f''_a)_{a \in a_0 + (-\eta', \eta')^k}$ is of class C^{r+1} -bounded.

By the second part of Remark 3.10, this perturbation is C^r -small on $a_0 + [-\eta', \eta']^k$, when n is large and η' -small, since by Facts 4.6 and 4.7, when n is large, $J_{a_0}^r(\Psi_a(P_{0a}) - S'_a)_a, J_{a_0}^r(D\Psi_a(T_{P_{0a}}\mathbb{T}'_{\mathbb{P}^0}))_a$ and $J_{a_0}^r(\frac{\lambda'_u(a)}{\lambda_u(a)})$ are small.

We recall that the perturbations done was localized in $\{(x, y) : y \in I_\epsilon\}$. Finally by using a bump function equal to 0 on the complement of $a_0 + [-\frac{3}{4}\eta', \frac{3}{4}\eta']^k$ and 1 over $a_0 + [-\frac{2}{3}\eta', \frac{2}{3}\eta']^k$ we display a wished family as equal to \hat{f}'' on $a_0 + [-\frac{2}{3}\eta', \frac{2}{3}\eta']^k$ and \hat{f}' on the complement of $a_0 + [-\frac{3}{4}\eta', \frac{3}{4}\eta']^k$. \square

4.4 Conclusion

Proof that Lemma 4.3 implies Theorem 2.3 in the case $\dim M = 2$. We observe that we can apply Proposition 3.11 to the conclusion of the latter claim. This shows the following.

For every $\hat{f}' \in \hat{U}$ of class C^∞ and $\eta > 0$ small, there exists $\eta' > 0$ so that for any $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$ there exists a C^r - η -small perturbation $f'' = \hat{f}^{a_0 \mathfrak{c}}$ of \hat{f}' satisfying:

- (i) For a periodic point $\bar{\mathbf{p}}^0 \in \mathfrak{A}^{(\mathbb{N})}$, for every $a \in [-\frac{2}{3}\eta', \frac{2}{3}\eta']^k + a_0$, the map $f_a^{''2n}|_{\mathbb{T}_{\bar{\mathbf{p}}^0}''(a)}$ is parabolic.
- (ii) $f_a''(x, y)$ is equal to $f'_a(x, y)$ if $a \notin a_0 + (-\eta', \eta')^k$ or $y \notin I_\epsilon$.
- (iii) The orbit of $\mathbb{T}_{\bar{\mathbf{p}}^0}''(a)$ is disjoint $\cup_{\mathfrak{c}' \in \mathfrak{C} \setminus \{\mathfrak{c}\}} \{y \in I_{\mathfrak{c}'}\}$.

We recall that $\mathfrak{C} = \{-1, 1\}^k$. Put $L := \eta' \mathbb{Z}^k \cap [-\alpha, \alpha]^k$ and for every $\mathfrak{c} \in \mathfrak{C}$, put $L_\mathfrak{c} = L \cap (\eta' \mathfrak{c} + 2\eta' \mathbb{Z}^k)$. We notice that:

- $L = \cup_{\mathfrak{c} \in \mathfrak{C}} L_\mathfrak{c}$,
- $[-\alpha, \alpha]^k \subset L + (-\frac{2}{3}\eta', \frac{2}{3}\eta')^k$,
- for every $\mathfrak{c} \in \mathfrak{C}$, for all $l, l' \in L_\mathfrak{c}$, the sets $l + (-\eta', \eta')^k$ and $l' + (-\eta', \eta')^k$ are disjoint.

Now we define the η - C^r -perturbation \hat{f}'' of \hat{f}' by:

$$\begin{cases} f_a''(x, y) = f_a^{a_0 \mathfrak{c}}(x, y) & \text{if } a \in a_0 + (-\eta', \eta')^k, a_0 \in L_\mathfrak{c} \text{ and } y \in I_\epsilon, \text{ for a certain } \mathfrak{c} \in \mathfrak{C}, \\ f_a''(x, y) = f'_a(x, y) & \text{otherwise.} \end{cases}$$

We notice that \hat{f}'' is η - C^r -close to \hat{f}' . Furthermore, by (ii) and (iii) these perturbations do not interfere on the parabolic fiber: for all $a \in a_0 + (-\frac{2}{3}\eta', \frac{2}{3}\eta')^k$ and $a_0 \in L_\mathfrak{c}$, the map f_a'' displays a periodic parabolic fiber. Furthermore, since $(a_0 + (-\frac{2}{3}\eta', \frac{2}{3}\eta')^k)_{a_0 \in L = \cup_{\mathfrak{c} \in \mathfrak{C}} L_\mathfrak{c}}$ is a finite covering of $[-\alpha, \alpha]^k$, Theorem 2.3 is proved for the reparametrized family $(f_{\alpha \cdot a})_{a \in [-1, 1]^k}$ and neighborhood $\{(f_{\alpha \cdot a})_a : \hat{f}' = (f'_a)_a \in \hat{U}\}$. \square

Proof of Theorem 2.3 in dimension $d \geq 3$. Let $(f_a)_a$ be the C^∞ -family of the annulus \mathbb{A} defined for the case $\dim M = 2$, with $f_a(x, y) = (f_y(x), \sigma(y))$. As for the proof of Theorem 2.2, §3.5, we unfold the dynamics in $\mathbb{A} \times [-2, 2]^{d-2}$ by defining for b small enough:

$$F_a : (x, y, z, (w_i)_i) \in \mathbb{A} \times [-2, 2]^{d-2} \mapsto (f_y(x), \sigma(y) - b \cdot z, b \cdot y, b \cdot (w_i)_i).$$

We recall that by Fact 3.16 we can embed the C^∞ -family $\hat{F} = (F_a)_a$ into any manifold of dimension M of dimension d .

As seen in §3.5, for b small enough, for every C^r -perturbation $\hat{F}' = (F'_a)_a$ of $(F_a)_a$, for every $a \in [-1, 1]^k$, the f_a -invariant circle bundle $(\mathbb{T}_{\bar{a}}(a))_{\bar{a} \in \mathfrak{A}^{\mathbb{N}}}$ persists as a F'_a -invariant circle bundle $(\mathbb{T}_{\bar{a}}(a))_{\bar{a} \in \mathfrak{A}^{\mathbb{Z}}}$. Assume that $(F'_a)_a$ is of class C^∞ . We notice that $(a, M) \mapsto (a, F'_a(M))$ is $(r+2)$ -normally hyperbolic at $(\cup_{a \in (-1, 1)^k} \{a\} \times \mathbb{T}_{\bar{a}}(a))_{\bar{a} \in \mathfrak{A}^{\mathbb{Z}}}$, and so by forced smoothness, $(\mathbb{T}_{\bar{a}}(a))_a$ is a C^{r+2} -family of circles, which depends continuously on $\bar{a} \in \mathfrak{A}^{\mathbb{Z}}$.

For every $a \in [-1, 1]^k$ and $\mathfrak{c} \in \mathfrak{C}$, let us continue to denote by $S'_{\mathfrak{c} a}$ the unique fixed point of F'_a in $\{y \in I_{\mathfrak{c}}\} \cap \{x \in [-1, 1]\}$. Let $\lambda_{uc}(a)$ be its weak unstable eigenvalue.

Similarly the maximal invariant of $\cup_{\mathfrak{b} \in \mathfrak{B}} [-3, 3] \times I_{\mathfrak{b}} \times [-2, 2]^{d-2}$ is the λ - C^r -parablender given by Example 1.27. For every $\underline{\mathfrak{b}} \in \mathfrak{B}^{\mathbb{Z}^-}$, let $W_{loc}^u(\underline{\mathfrak{b}}; F'_a)$ be the hyperbolic continuation of the local unstable manifold $W_{loc}^u(\underline{\mathfrak{b}}; f_a) \times \{0\}$ for $(f_a, 0)$. By Example 1.27 it holds:

Fact 4.8. *For a small C^r -neighborhood \hat{U} of \hat{F} , there exists $\alpha > 0$, so that for every n large enough, for every $\hat{F}' \in \hat{U}$, for every $a_0 \in [-\alpha, \alpha]^k$, for every $\mathfrak{c} \in \mathfrak{C}$, there exist $\underline{\mathfrak{b}} = (\mathfrak{b}_i)_{i \leq 0} \in \mathfrak{B}^{\mathbb{Z}^-}$ and a C^r -curve $(S'_{0 a})_a \in (\mathbb{T}'_{\underline{\mathfrak{b}}, \mathfrak{c}^{\mathbb{N}}}(a))_a$ and $(P_{\mathfrak{c} a})_a \in (W_{loc}^{ss}(S'_{\mathfrak{c} a}; F'_a))_a$ satisfying:*

$$J_{a_0}^r(S'_{0 a})_a = J_{a_0}^r(P_{\mathfrak{c} a})_a \quad \text{and} \quad \lim_{n \rightarrow \infty} J_{a_0}^{r-1}(\|Df_a^n|_{T_{S'_{-n a}} S'_{-n a}} \mathbb{T}_{\sigma^{-n}(\underline{\mathfrak{b}}, \mathfrak{c}^{\mathbb{N}})}\|^{-\frac{1}{n}})_a = J_{a_0}^{r-1}(\lambda_{uc}(a))_a,$$

with $S'_{-n a}$ the preimage by F_a^n of $S'_{0 a}$.

Given n large, $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$, we denote by $\overleftarrow{\mathfrak{p}}^0(n, a_0, \mathfrak{c}, \hat{F}')$ the $2n$ periodic point $(\mathfrak{c} \cdots \mathfrak{c} \cdot \mathfrak{b}_{-n} \cdots \mathfrak{b}_{-1})^{\mathbb{Z}}$ of $\mathfrak{A}^{\mathbb{Z}}$, where $\mathfrak{b}_{-n} \cdots \mathfrak{b}_{-1}$ are the n last symbols of $\underline{\mathfrak{b}}$ given by fact 4.8.

By merging the Proof of Lemma 4.3 and Claim 3.19, it comes:

Lemma 4.9. *For every $\eta > 0$ and for every $\hat{f}' \in \hat{U}$ of class C^∞ , there exist $n \geq 1$ and $\eta' > 0$ so that for every $a_0 \in [-\alpha, \alpha]^k$ and $\mathfrak{c} \in \mathfrak{C}$ there exists a C^r - η -small perturbation \hat{F}'' of \hat{F}' supported in $([-\eta', \eta']^k + a_0) \times \mathbb{T} \times I_{\mathfrak{c}} \times [-2, 2]^{d-2}$, of class C^{r+1} and satisfying the following property.*

With $\overleftarrow{\mathfrak{p}}^0 = \overleftarrow{\mathfrak{p}}^0(n, a_0, \mathfrak{c}, \hat{F}')$ the above $2n$ -periodic point of $\mathfrak{A}^{\mathbb{N}}$, for every $a \in [-\frac{2}{3}\eta', \frac{2}{3}\eta']^k + a_0$, the map F''_a displays a semi-parabolic $2n$ -periodic, point in $\mathbb{T}''_{\overleftarrow{\mathfrak{p}}^0}(a)$.

This Lemma implies Theorem 2.3 for $\dim M = d$, by exactly the same argument as Lemma 4.3 implied Theorem 2.3 for $\dim M = 2$ (except that Prop. 3.11 is changed to Prop. 3.21). \square

5 Perturbation of families of parabolic circle maps to constant rotations

This section is devoted to the proof of Theorem 1.4. Let $k \in \mathbb{N}$ and let $V \subset \mathbb{R}^k$ be an open subset. Let $(g_a)_{a \in V}$ be a C^∞ -family of circle diffeomorphisms so that for every $a \in V$ the map g_a is parabolic. Given any $V' \Subset V$, we want to find a C^∞ -perturbation $(g'_a)_{a \in V'}$ of $(g_a)_{a \in V}$ so that g'_a displays a Diophantine rotation number which does not dependent on $a \in V'$.

We will work with the coordinates given by a one-point compactification $\hat{\mathbb{R}}$ of \mathbb{R} . Hence given $a < b \in \mathbb{R}$, we will denote by (a, b) the segment of $\mathbb{R} \subset \hat{\mathbb{R}}$ and by (b, ∞, a) the arc of $\hat{\mathbb{R}}$ containing ∞ and with endpoints $\{a, b\}$.

Sketch of proof The proof of the theorem is done by the following steps:

1. First we show that after a small perturbation, we can assume moreover that:
 - (a) g_a fixes 0, satisfies $D^2g_a(0) = 2$, for every $a \in V$, and it holds g_a sends $\{1/2, 1, -1\}$ to $\{1, -1, -1/2\}$.
 - (b) The restriction g_a to a small connected neighborhood N of 0 is the time one flow of a polynomial flow X_a .
 - (c) The map $(x, a) \mapsto g_a(x)$ is of class C^∞ .
2. Hypothesis (b) will enable us to show that $g_a|(-1, 1)$ is the time one of a flow given by a vector field $X_a|_{\hat{\mathbb{R}} \setminus \{-1\}}$ depending smoothly on the parameter a . We look at a family of perturbations $(g_a)_\eta$ defined by:
 - (a) g_a_η is equal to g_a on $(1, \infty, -1)$,
 - (b) g_a_η is equals to the time one map of the flow defined by the field $X_a + \rho(x)\eta^2$ with $\rho \in C^\infty(\mathbb{R}, [0, 1])$ equal to 1 on a neighborhood of 0 and with support in $(-1/2, 1/2)$.
 Furthermore, we show that in some coordinates (given by the two canonical extensions of X_a over $(1, \infty, -1)$) the first return map G_{a_η} of g_{a_η} in $(1, \infty, -1)$ is of the form $G_{a_0} + \omega_a^+(\eta)$, with $(a, \eta) \mapsto \omega_a^+(\eta)$ of class C^∞ .
3. We show the existence of a C^∞ -family $(\Omega_a)_a$ of functions $\Omega_a \in C^\infty([0, \infty), \mathbb{R})$ such that $\Omega_a(0) = 1/\pi$ and $\omega_a^+(\eta) = \Omega_a(\eta)/\eta \pmod 1$ for every a . This is done by formulating $\omega_a^+(\eta)$ as an integral.
4. We use the following KAM's Theorem of Herman-Yoccoz:

Theorem 5.1 (Herman-Yoccoz). *For every $\beta \in \mathbb{R}$ diophantine, let V_β be the set of circle diffeomorphisms in $\text{Diff}^\infty(\mathbb{T})$ whose rotation number is β . Then V_β is a smooth submanifold of codimension 1. Moreover, for every $f \in V_\beta$, the family $(f + b)_{b \in \mathbb{R}}$ is transverse to V_β at $b = 0$.*

Proof. By Yoccoz' Theorem 1.1, we can assume that f is the rotation of angle β . Then the Theorem is stated in Remark 3.1.3 of [Bos85]. \square

Then given a diophantine number β , we can define implicitly an arbitrarily C^∞ -small function $a \mapsto \eta(a)$ (for the compact-open topology) so that the return map $G_{a_{\eta(a)}}$ displays the rotation number β for every $a \in V'$.

This implies that the rotation number of $g_{a_{\eta(a)}}$ is of the form $1/(N + \beta)$, and so is diophantine as well. In other words, with $g'_a := g_{a_{\eta(a)}}$, the family $(g'_a)_a$ satisfies the wished property.

Step 1 : Setting a) Let $x_a \in \hat{\mathbb{R}}$ be the fixed point of g_a for $a \in V$. As the map g_a is parabolic, $\partial_x g_a(x_a) = 1$ and $\partial_x^2 g_a(x_a) \neq 0$. Hence, x_a can be defined (locally) implicitly by:

$$\Psi(x, a) := \partial_x g_a(x) - 1 \quad \Psi(x_a, a) = 0$$

Hence by the implicit function theorem, the map $a \mapsto x_a$ is of class C^∞ . Thus by a smooth coordinate change, we can assume that $x_a = 0$ for every $a \in V$.

Again, by shrinking slightly V , we can conjugate the dynamics by the Moebius function $x \mapsto \frac{2 \cdot x}{D^2 g_a(x_a)}$ for every $a \in V$. We notice that for every $a \in V$ it holds:

$$g_a(0) = 0 \quad , \quad Dg_a(0) = 1 \quad , \quad D^2 g_a(0) = 2 \quad .$$

To accomplish this step it suffices to conjugate g_a by a smooth family of diffeomorphisms, equal to the identity on a neighborhood of 0 and which sends $\{1/2, g_a(1/2), g_a^2(1/2), g_a^3(1/2)\}$ to $\{1/2, 1, -1, -1/2\}$.

b) Let $\mathfrak{G}_d = \{\sum_{j=2}^d \frac{a_j}{j!} x^j : (a_j)_{j=2}^d \in \mathbb{R}^{d-2}\}$. Given $\mathfrak{p} \in \mathfrak{G}_d$, let $\phi_{\mathfrak{p}}$ be the time one flow of the vector field $x + \mathfrak{p}(x)$. Let $J_0^d \phi_{\mathfrak{p}}$ be the C^d -jet of $\phi_{\mathfrak{p}}$:

$$J_0^d \phi_{\mathfrak{p}} = \sum_{i=0}^d \frac{\partial^i \phi_{\mathfrak{p}}}{i!} x^i = x + \sum_{i=2}^d \frac{\partial^i \phi_{\mathfrak{p}}}{i!} x^i \quad .$$

In other words, $J_0^d \phi_{\mathfrak{p}}$ belongs to the space G_d of C^d -jets of parabolic maps at 0:

$$G_d = \{x + \sum_{j=2}^d \frac{a_j}{j!} x^j + o(x^d) : (a_j)_{j=2}^d \in \mathbb{R}^{d-2}\} \quad \text{for } d \geq 0 \quad .$$

Proposition 5.2. *The map $\mathfrak{p} \in \mathfrak{G}_d \mapsto \phi_{\mathfrak{p}} \in C^\infty(\mathbb{R}, \mathbb{R})$ is smooth.*

Moreover, the following map is a diffeomorphism onto its image:

$$\Psi : \mathfrak{p} \in \mathfrak{G}_d \mapsto J_0^d \phi_{\mathfrak{p}} \in G_d \quad .$$

Proof. The first statement of this proposition is a simple consequence of the Cauchy-Lipschitz Theorem. The second part of the proposition involves the Lie group theory. Indeed, the C^d -jet space G_d endowed with the composition rules is a lie group. Moreover it satisfies:

Fact 5.3. *The group G_d is connected, simply connected and nilpotent.*

Proof. The group G_d is homeomorphic to \mathbb{R}^{d-1} , hence it is connected and simply connected. Let $G_d^{(s)} := \{\phi \in G_d : \phi(x) = x + O(x^{s+2})\}$ for $s \geq 0$. A computation gives $[G_d, G_d^{(s)}] \subset G_d^{(s+1)}$. As $G_d^{(d-1)}$ is trivial, G_d is nilpotent with rank $\leq d-1$. \square

We notice that \mathfrak{G}_d is the Lie algebra of the group G_d . Moreover, $\phi_{\mathfrak{p}}$ is the image by the exponential map \exp of $\mathfrak{p} \in \mathfrak{G}_d$. Indeed, if $\phi_{\mathfrak{p}}^t = \exp(t \cdot \mathfrak{p})$ and so $\phi_{\mathfrak{p}} = \phi_{\mathfrak{p}}^1$, it holds:

$$\phi_{\mathfrak{p}}^{t+dt} = \phi_{\mathfrak{p}}^{dt} \circ \phi_{\mathfrak{p}}^t = (id + dt \cdot \mathfrak{p}) \circ \phi_{\mathfrak{p}}^t + o(dt)$$

Hence the proposition is implied by the fact that given a Lie group which is simply connected and nilpotent, the exponential map is an analytic diffeomorphism from its Lie algebra onto its image [CG90, pp. 13, Thm 1.2.1]. \square

Corollary 5.4. *There exists a C^∞ -map $a \mapsto X_a \in \mathfrak{G}_d$ so that $J_0^d \phi_{X_a} = J_0^d g_a$ for every a .*

Let us fix a bump function ρ equal to 1 on $[-1/2, 1/2]$ and with support in $(-1, 1)$, and for $\epsilon > 0$ small, put:

$$\tilde{g}_a : x \mapsto \rho(x/\epsilon) \cdot \phi_{X_a}(x) + (1 - \rho(x/\epsilon)) \cdot g_a(x) \quad .$$

Lemma 5.5. *For $\epsilon > 0$ sufficiently small and d sufficiently large, the family $(\tilde{g}_a)_{a \in V}$ is C^∞ -close to $(g_a)_{a \in V}$ and satisfies Conditions (a) – (b) – (c).*

Proof. We have $g_a(x) - g'_a(x) = \rho((x - \tilde{x}_a)/\epsilon)(g_a(x) - \phi_{X_a}(x))$. For every $j \leq d$, we notice that $(a, x) \in V \times [-\epsilon, \epsilon] \mapsto g_a(x) - \phi_{X_a}(x)$ has its j -derivative which is small w.r.t. ϵ^{d-j} . On the other hand, the j^{th} -derivatives of $\rho(x/\epsilon)$ is dominated by ϵ^{-j} . Hence by Leibnitz formula, the C^d -norm of $(a, x) \in V \times (-\epsilon, \epsilon) \mapsto g_a(x) - \tilde{g}_a(x)$ is small when ϵ is small. \square

Hence we can suppose that $(g_a)_{a \in V}$ satisfies $(a - b - c)$. Furthermore:

Claim 5.6. *It holds $X_a(x) = x^2 + O(x^3)$ for every a .*

Proof. $J_0^d g_a^2(x) - J_0^d g_a(x)$ is equivalent to both $\frac{D^2 g_a(0)}{2} x^2 = x^2$ as $x \rightarrow 0$ and $X_a(x)$. \square

Step 2 : Definition of the 2-parameters family $(g_a)_a$ and uniform bound on the first return map Let $(g_a)_a$ be a family of diffeomorphisms satisfying $(a - b - c)$. We recall that for $x \in N \cap g_a^{-1}(N)$ it holds:

$$(1) \quad X_a(x) = Dg_a^{-1} \circ X_a \circ g_a(x) .$$

Hence we can extend X_a by pushing forward and backward it, in order to assume that $N = (-1, 1)$.

Let ρ be a bump function equal to 1 on a neighborhood of 0 and with support in $(-1/2, 1/2)$. For $\eta > 0$ small, we define:

$$X_{a\eta} : x \in [-1/2, 1/2] \mapsto X_a(x) + \eta^2 \cdot \rho(x) .$$

We observe that $[1, \infty, -1]$ is sent by g_a^{-1} onto $[1/2, 1) \subset N$, and $(1, \infty, -1]$ is sent by g_a onto $(-1, -1/2] \subset N$. For every $x \in [-1, 1/2]$, let $\phi_{a\eta}^1$ be the time one flow of $X_{a\eta}$.

Note that the map $\phi_{a\eta}^1$ is equal to f_a on the intersection of a neighborhood of $\{1/2\} \cup \{-1\}$ with its domain. This implies:

Claim 5.7. *The following family $(g_a)_a$ of C^∞ -dynamics $g_{a\eta}$ is well defined and smooth:*

$$g_{a\eta} : x \mapsto \begin{cases} \phi_{a\eta}^1(x) & \text{if } x \in [-1, 1/2] , \\ g_a(x) & \text{if } x \in (1/2, \infty, -1) . \end{cases}$$

For $\eta > 0$, we are going to study the first return map $T_{a\eta}$ of $g_{a\eta}$ into $[1, \infty, -1]$. We recall that $g_{a\eta}(1) = -1$.

For this end we shall work with two different possible extensions of X_a on $(1, \infty, -1)$. Let:

$$\begin{cases} X_a^+ := x \in [1, \infty, -1] \mapsto Dg_a \circ X_a \circ g_a^{-1}(x) , \\ X_a^- := x \in (1, \infty, -1] \mapsto Dg_a^{-1} \circ X_a \circ g_a(x) . \end{cases}$$

In general X_a^+ and X_a^- are different.

As X_a is equal to $X_{a\eta}$ on $\cup_{i \in \{-1, 0, 1\}} g_a^i([1, \infty, -1]) = [1/2, \infty, -1/2]$, it holds :

Fact 5.8. *The vector field X_a^+ extends smoothly $X_{a\eta}$ to a vector field denoted by $X_{a\eta}^+$ on $\hat{\mathbb{R}} \setminus \{-1\}$. Also, X_a^- extends smoothly $X_{a\eta}$ to a vector field denoted by $X_{a\eta}^-$ on $\hat{\mathbb{R}} \setminus \{+1\}$.*

To study the first return map in $[1, \infty, -1]$ by $g_{a\eta}$ we consider the coordinates C_a^+ and C_a^- on $[0, 1] \rightarrow [1, \infty, -1]$ so that³:

- $C_a^+(t)$ is the image of 1 by the time t flow of X_a^+ ,

³In this definition, we considered the extensions of $X_a^+|[1, \infty, -1]$ and $X_a^-|(1, \infty, -1]$ to $[1, \infty, -1]$.

- $C_a^-(t)$ is the image of 1 by the time t flow of X_a^- .

Note that both maps C_a^\pm send $[0, 1]$ onto $[1, \infty, -1]$. Remark also that both C_a^\pm do not depend on η since X_a^\pm does not depend of η on $[1, \infty, -1]$. We define the following coordinates change:

$$\Phi_a: [0, 1] \mapsto (C_a^-)^{-1} \circ C_a^+(t) \in [0, 1] .$$

Let $T_{a\eta}^- := (C_a^-)^{-1} \circ T_{a\eta} \circ C_a^-$ be the first return map $T_{a\eta}$ seen in the coordinates C_a^- .

Let $\omega_a(\eta) = g_{a\eta}^N(-1) \in [1, \infty, -1]$ be the first return of -1 into $[1, \infty, -1]$. Let $\alpha_a(\eta) \in [1, \infty, -1]$ be the preimage $\alpha_a(\eta) := (g_{a\eta}^N)^{-1}(1) \in [1, \infty, -1]$.

Let $\alpha_a^\pm(\eta)$ and $\omega_a^\pm(\eta)$ be the preimages of $\alpha_a(\eta)$ and $\omega_a(\eta)$ by C_a^\pm . We observe that:

$$T_{a\eta}^-(1) = \omega_a^-(\eta) \quad \text{and} \quad T_{a\eta}^-(\alpha_a^-(\eta)) = 0 .$$

We depict in figure 2 the notations involved in this renormalization.

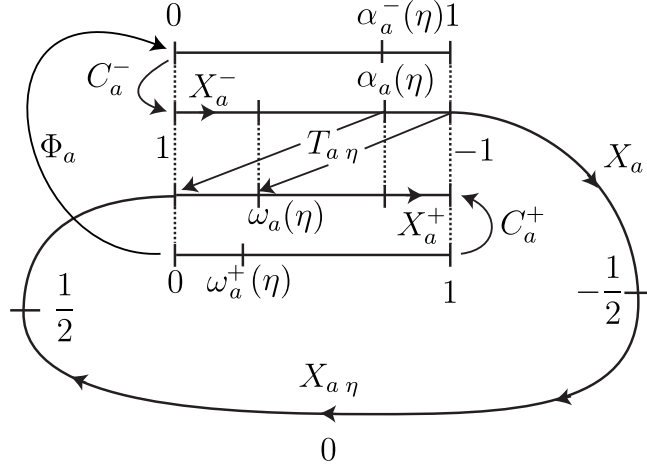


Figure 2: Notations for the parabolic renormalization.

Claim 5.9. *The first return map $T_{a\eta}^-$ satisfies:*

$$\begin{cases} T_{a\eta}^-(s) = \Phi_a(s + 1 - \alpha_a^-(\eta)) = \Phi_a(s + \omega_a^+(\eta)) & \text{if } s < \alpha_a^-(\eta) , \\ T_{a\eta}^-(s) = \Phi_a(s - \alpha_a^-(\eta)) = \Phi_a(s + \omega_a^+(\eta) - 1) & \text{if } s > \alpha_a^-(\eta) . \end{cases}$$

Proof. First let us notice that by commutativity of the flow $X_{a\eta}$ with $g_{a\eta}^N$, it holds:

$$1 - \alpha_a^-(\eta) = \omega_a^+(\eta) .$$

If $s > \alpha_a^-(\eta)$, then the first return of $x = C_a^-(s)$ in $[1, \infty, -1]$ is the image by the flow X_a^+ of 1 after a time $s - \alpha_a^-(\eta)$. In the coordinate C_a^+ , it is $s - \alpha_a^-(\eta)$. In the coordinate C_a^- , it is $\Phi_a(s - \alpha_a^-(\eta))$.

If $s < \alpha_a^-(\eta)$, then the first return of $x = C_a^-(s)$ in $[1, \infty, -1]$ is the image by the flow X_a^+ of $\omega_a(\eta)$ after a time s . In the coordinate C_a^+ , it is $\omega_a^+(\eta) + s$. In the coordinate C_a^- , it is $\Phi_a(\omega_a^+(\eta) + s)$. \square

Claim 5.10. *After gluing the endpoints of $[0, 1]$ by the translation by 1, the map $T_{a, \eta}^-$ is a smooth map of the circle \mathbb{R}/\mathbb{Z} , and the family $(T_{a, \eta}^-)_{a, \eta}$ is of class C^∞ .*

Proof. It is classical [Yoc95] that the first return map $g_{a, \eta}$ into $[1, g_{a, \eta}(1)]$ is projected to a smooth maps of the circle obtained by gluing the endpoints of $[1, g_{a, \eta}(1)]$ thanks to $g_{a, \eta}$. Seen in the chart C_a^- , this corresponds to glue the endpoints of $[0, 1]$ thanks to the translation by $+1$. In this specific parabolic context, this map was called "the essential map" in [ST00]. \square

With the conjugacy $s \mapsto s + \omega_a^+(\eta)$ we obtain:

Corollary 5.11. *The first return map $T_{a, \eta}$ of is smoothly conjugated to:*

$$R_{a, \eta}: s \in \mathbb{R}/\mathbb{Z} \mapsto \Phi_a(s) + \omega_a^+(\eta) \in \mathbb{R}/\mathbb{Z}$$

Step 3 We shall study the derivative of $\omega_a^+(\eta) \pmod 1$ with respect to η is large when $\eta \rightarrow 0$. We recall that the time needed for the flow $X_{a, \eta}$ to go from -1 to 1 , is:

$$\tau_a(\eta) := \int_{[-1, 1]} \frac{1}{X_{a, \eta}} d\text{Leb}.$$

We notice that $\omega_a^+(\eta) + \tau(\eta) = N \in \mathbb{Z}$ and so:

$$\omega_a^+(\eta) = -\tau_a(\eta) \pmod 1.$$

By Claim 5.6, there exists $X_{a, \eta}^1 \in C^\infty([-1, 1], \mathbb{R})$:

$$X_{a, \eta}(x) = x^2 X_{a, \eta}^1 + \eta^2.$$

As g_a has a unique fixed point at 0, the field $X_{a, 0}^1$ is positive on $[-1, 1]$, with value 1 at 0 by Claim 5.6. Thus for η small, there exists $C > 0$ such that:

$$(2) \quad X_{a, \eta}^1(0) = 1 \quad \text{and} \quad X_{a, \eta}^1 \geq C > 0.$$

In these notations, it holds:

$$\eta \cdot \tau_a(\eta) = \int_{[-1, 1]} \frac{\eta}{s^2 \cdot X_{a, \eta}^1(s) + \eta^2} ds.$$

We put $s = \eta \cdot t$ and we have:

$$\eta \cdot \tau_a(\eta) = \int_{-1/\eta}^{1/\eta} \frac{1}{1 + t^2 X_{a, \eta}^1(\eta t)} dt, \quad \forall \eta \neq 0.$$

Let $\Psi(t, a, \eta) := 1 + t^2 X_{a, \eta}^1(\eta t)$.

Lemma 5.12. *For every $n \geq 0$, there exists $C_n > 0$ so that for every $a \in V'$ and η small:*

$$\left| \partial_{a, \eta}^n \frac{1}{\Psi(t, a, \eta)} \right| \leq \frac{C_n}{1 + t^2}$$

Proof. The case $n = 0$ is an immediate consequence of Inequality (2). Let $n \geq 0$ and assume by induction that Lemma 5.12 holds for every $k \leq n$. By Leibniz formula applied to Ψ/Ψ , it holds:

$$\sum_{k=0}^{n+1} C_{n+1}^k \partial_{a, \eta}^k \frac{1}{\Psi(t, a, \eta)} \cdot \partial_{a, \eta}^{n+1-k} \Psi(t, a, \eta) = 0$$

$$\Rightarrow \partial_{a\eta}^{n+1} \frac{1}{\Psi(t, a, \eta)} = -\frac{1}{\Psi(t, a, \eta)} \sum_{k=0}^n C_{n+1}^k \partial_{a\eta}^k \frac{1}{\Psi(t, a, \eta)} \cdot \partial_{a\eta}^{n+1-k} \Psi(t, a, \eta) .$$

It is easy to see that $\partial_{a\eta}^{n+1-k} \Psi(t, a, \eta)$ is bounded by a certain Ct^2 . Hence the induction hypothesis gives:

$$\left| \partial_{a\eta}^{n+1} \frac{1}{\Psi(t, a, \eta)} \right| \leq C_0 \frac{1}{1+t^2} \sum_{k=0}^n Ct^2 C_k \frac{1}{1+t^2}$$

Hence the above sum is bounded from above by $C_{n+1} \frac{1}{1+t^2}$ for $C_{n+1} \in \mathbb{R}$ independent of $t, a \in V$ and η small. \square

We notice that by the dominate function theorem, the following function is of class C^∞ :

$$(a, \eta) \mapsto \eta \cdot \tau_a(\eta) .$$

Moreover it holds:

$$\lim_{\eta \rightarrow 0} \eta \cdot \tau_a(\eta) = \int_{\mathbb{R}} \frac{1}{\Psi(t, a, 0)} dt = \int_{\mathbb{R}} \frac{1}{1+t^2} dt = \pi .$$

Consequently it holds:

Fact 5.13. *There exists a C^∞ -family $(\Omega_a)_a$ of C^∞ -functions $\Omega_a \in C^\infty([0, \infty), \mathbb{R})$ so that*

$$\omega_a^+(\eta) = \Omega_a(\eta)/\eta \mod 1 .$$

Proof. Indeed from the above discussion, the function $\Omega_a(\eta) := -\eta \cdot \tau_a(\eta)$ depends smoothly on $a \in \mathbb{R}, \eta \geq 0$. \square

Step 4 We recall that the coordinate changes Φ_a is projected to a C^∞ -diffeomorphism of the torus \mathbb{R}/\mathbb{Z} , and $(\Phi_a)_a$ is of class C^∞ .

Here is an imediate consequence of Herman-Yoccoz' Theorem 5.1:

Corollary 5.14. *For every Diophantine number β , there exists a C^∞ -function $a \in V \mapsto B(a)$ so that $\Phi_a + B(a) \in \mathbb{R}/\mathbb{Z}$ has rotation number equals to β .*

Hence it remains to solve implicitly:

$$\omega_a^+(\eta(a)) = B(a) \mod 1 \iff \frac{\Omega_a(\eta(a))}{\eta(a)} = B(a) \mod 1 .$$

As $\Omega_a(0) = \pi$, for every $a_0 \in V$, there exists $\eta(a_0)$ arbitrarily small so that

$$\frac{\Omega_a(\eta(a_0))}{\eta(a_0)} = B(a_0) \mod 1 .$$

Note that the following derivative is large for η small (since $\Omega_a(0) = \pi$ and $\partial_\eta \Omega_a(0)$ is bounded).

$$\partial_\eta \left(\frac{\Omega(\eta)}{\eta} \right) = \frac{\eta \cdot \partial_\eta \Omega_a(\eta) - \Omega_a(\eta)}{\eta^2} \sim -\frac{\pi}{\eta^2} .$$

Hence, the implicit function theorem enables us to conclude that for every $V' \Subset V$ there exists a small smooth function $a \in V' \mapsto \eta(a)$ so that $R_{a \eta(a)}$ has rotation number equal to β . Then, $g_{a \eta(a)}$ has a rotation number of the form $1/(N + \beta)$ which is diophantine as well. Note that $(g_{a \eta(a)})_{a \in V'}$ is C^∞ -close to $(g_a)_{a \in V'}$. Thus Theorem 1.4 is proved. \square

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